Suppose we have data on N people collected at time period b. Each of these people began to be unemployed at time a_i. When we observe them, some people are now employed; denote the duration of their unemployment D_i. For the people that remained unemployment, we only know that their unemployment spell is at least $b - a_i$ long. We as researchers are interested in the following question: how does the amount of time someone remains unemployed D_i^* impact their wage w_i^* when they do become employed? To answer this question, we specify the following two equations:

$$log(D_i^*) = \beta_0 + \beta_1 Z_i + u_i \tag{1}$$

$$log(\mathrm{W}_i^*) = \theta_0 + \theta_1 log(\mathrm{D}_i^*) + v_i \tag{2}$$

For parts (a)-(c), assume for simplicity that we actually observe duration D_i^* and wages W_i^*. For part (d), we only observe the full duration D_i if the person became employed by time b and we only observe wage W_i if the person is employed at time b.

Finally, assume that:

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} | Z_i \sim Normal(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_u^2 & \sigma_{u,v} \\ \sigma_{u,v} & \sigma_v^2 \end{pmatrix})$$

Assume that Z_i is independent of both u_i and v_i. Z_i is correlated with X_i.

1. Suppose we observe D_i^* and W_i^*. Suppose we estimate Equation 1 using ordinary least squares (OLS). What are the probability limits of the resulting OLS estimates $\hat{\beta}_0$ and $\hat{\beta}_1$?

2. Suppose we observe D_i^* and W_i^*. Now, we estimate **equation (2)** using **ordinary least squares** (OLS). What are the probability limits of the resulting OLS estimates $\widehat{\theta_0}$ and $\widehat{\theta_1}$ and when are these estimates consistent?

3. Suppose we observe D_i^* and W_i^*. Propose an instrumental variables (IV) estimator for **equation (2)** and prove that your IV estimator is consistent for true parameters θ_0 and θ_1.

4. **No longer assume** that we observe D_i^* and W_i^*. We assume that we know σ_u. Write down the maximum likelihood estimator for **equation (1)** and the asymptotic distribution of $\widehat{\beta_0}$ and $\widehat{\beta_1}$ estimated using maximum likelihood.