PROBLEM # 1:Multiple Choice — Choose the correct answer. Each multiple choice question is worth (5) points. The problems with * could have more than one answer. The first correct answer gives 3 points; Any additional correct answer gives you 2 points, but any additional wrong answer will cost you 2 points.

1.1* ________ Consider Y_1, Y_2, ... ,Y_n are i.i.d. random variables with $n > 4$ and T is the sample mean $\overline{Y}$. Also let $W = \frac{1}{4}\sum_{i=1}^{4} Y_i$. Then (A) $\mathrm{E}(Y_1 + Y_2|W) = W$ (B) $E(Y_1 + W|W) = E(Y_3 + Y_4|W)$ (C) $\mathrm{E}(W|T) = E(Y_1|T)$ (D) none of the above.

1.2 ________ Let X and Y be 2 random variables: $X \sim Exp(\lambda = 1)$ and $Y \sim Exp(\lambda = 0.5)$. Then we know that $\mathrm{Cov}(X, Y)$ is (A) ≥ -1 (B) ≥ 0 (C) ≤ 1 (D) ≤ 2 (E) None of the above is correct.

1.3* ________ Let X be a continuous random variable with pdf, $f_X(x)$, and $f_X(t + 3) = f_X(3 - t)$ for all $t > 0$. Which of the following statement is correct? (A) $P(X > 3) = 0.5$ (B) $P(X \leq 3) > P(X \geq 3)$ (C) $\mathrm{E}(X) = 3$ (D) $\int_3^\infty f_X(x)dx < P(X < 3)$.

1.4* ________ Let X_1, X_2, ... ,X_n be i.i.d random variables with the density $f_X(x; \theta)$. Which of the following statements are correct?

(A) If $\hat{\theta}$ is an unbiased estimator of θ, then $2\hat{\theta}$ is unbiased for 2θ.

(B) If $\hat{\theta}$ is an unbiased estimator of θ, then $\hat{\theta}^2$ is unbiased for θ^2.

(C) If $\hat{\theta}$ is a consistent estimator of θ, then $\hat{\theta}^2$ is a consistent estimator of θ^2.

(D) If $T = (T_1, T_2)$ provides a sufficient statistic for θ, then $W = (T_1+T_2, T_1-T_2)$ also provides a sufficient statistic for θ.

PROBLEM # 2: Suppose $X_1, X_2, \dots, X_n$ are i.i.d Exponential(θ), $Y_1, Y_2, \dots, Y_n$ are i.i.d Exponential(3θ), and they are mutually independent.

2.1 Find a sufficient statistic that consists of all X and Y observations.

2.2 Use observations from both X and Y to obtain the MLE of θ.

2.3 Please provide a 95% confidence interval for θ and justify your answer (feel free to assume that n is large).

PROBLEM # 3: Let X_1, X_2, X_3 follow a multinomial distribution with parameters n and $\pi = (\pi_1, \pi_2, \pi_3)$.

3.1 Consider the scenario that $\pi_1 = \pi_2 = 1 - \theta$, and $\pi_3 = \theta$, please describe the testing procedure for the most powerful test with the significance level α when we test $H_0 : \theta = 0.2$ vs $H_a : \theta = 0.4$ (feel free to assume that n is large for your testing procedure).

3.2 Now to check the assumed structure, we let the H_0 be $\pi_1 = \pi_2 = 1 - \theta$, and $\pi_3 = \theta$, and H_a be that the H_0 is incorrect. Please provide a test statistic and the corresponding testing procedure for a generalized likelihood ratio test with the significance level α for this problem. Feel free to use the notation of $\chi^2_q(\alpha)$ as the $1 - \alpha$ quantile of the Chi-square distribution with the degree of freedom q.

PROBLEM # 4: Suppose we observe 3 sets of independent random variables: $X_1,\ X_2,\ \ldots\ ,X_n \sim$ N$(\theta_x,\ \sigma^2)$; $Y_1,\ Y_2,\ \ldots\ ,Y_n \sim$ N$(\theta_y,\ \sigma^2)$; and $Z_1,\ Z_2,\ \ldots\ ,Z_n \sim$ N$(\theta_x+\theta_y,\ \sigma^2)$, which are also independent of each other. Let $S_x^2 = \sum_{i=1}^n (X_i - \overline{X})^2/(n-1)$, $S_y^2 = \sum_{i=1}^n (Y_i - \overline{Y})^2/(n-1)$,and $S_z^2 = \sum_{i=1}^n (Z_i - \overline{Z})^2/(n-1)$, where $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ are respectively the sample means for the X's, Y's and Z's. Also we let $S^2 = (1/3)(S_x^2 + S_y^2 + S_z^2)$.

4.1 Use all observations to obtain the MLE of σ^2.

4.2 What is the asymptotic distribution of the MLE, $\hat{\sigma}^2_{MLE}$? If you have a problem to find the answer, find the asymptotic distribution of $S^2 = (1/3)(S_x^2 + S_y^2 + S_z^2)$ to obtain maximum of 4 points instead (you do not need to do this if you obtain the answer for the MLE).

4.3 Let

$$T = \frac{\overline{X} + \overline{Y} - \overline{Z}}{S}.$$

One argues that $a \times T$, where a is a constant, follows a t-distribution. If you agree with the statement, please provide the constant a and an appropriate degree of freedom, and justify this statement. Provide justification why this is incorrect if you don't agree with the statement.