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Assignment 2 (Due 7 May))

1. Suppose that the variables y_t, w_t, z_t, are observed over T time periods ($t = 1,\ldots,T$) and it is thought that $E[y_t]$ depends linearly on w_t (without a constant) but not on z_t for the first T_1 periods, and $E[y_t]$ depends linearly on z_t (without a constant) but not on w_t for the last T_2 periods, with $T_1 + T_2 = T$. Write down a linear regression model that could be used to test this hypothesis, and specify the nature of the hypothesis in terms of the coefficients of the model. [Hint: try using four regressors.]

2. For the simple "location" model

$$y_t = \mu + u_t, \quad t = 1,...,T$$

(a) Suppose that $u_t = \varepsilon_t + \rho z$ where the ε_t are T i.i.d. $(0, \sigma_\varepsilon^2)$ random variables and z is a random variable, independent of each of the ε_t and with $E[z] = 0$, $var[z] = 1$. Find the $T \times T$ covariance matrix of the u_t. Show by means of Kruskal's theorem or otherwise that the OLS estimator of the parameter μ is BLU.

(b) Find the variance of the OLS estimator for the set up in (a) and use this to show that this estimator is consistent if $\rho = 0$ but inconsistent if $\rho \neq 0$. (To demonstrate consistency here show that the variance of the estimator tends to zero as $T \to \infty$, and for inconsistency show that this does not happen.) Does this make sense intuitively?

(c) Suppose that $u_t = \varepsilon_t + \theta\varepsilon_{t-1}$, where the ε_t are i.i.d. $(0, \sigma_\varepsilon^2)$. Find the covariance matrix of the u_t, and use this to show that the condition of Kruskals's theorem is satisfied, except for the first and last rows of the matrices appearing in the condition. What can we deduce from this ?

3. Consider the autoregessive model

$$y_t = \beta y_{t-1} + u_t, \quad t = 1,...,T$$

$$u_t = \varepsilon_t + \theta\varepsilon_{t-1}$$

Where the ε_t are independently and identically distributed with $E[\varepsilon_t] = 0$, $E[\varepsilon_t^2] = \sigma_\varepsilon^2$, and $y_0 = 0 = \varepsilon_0$.

(a) Suppose that $\theta = 0$. Is the OLS estimator, $b = \sum_{i=2,T} y_t y_{t-1} / \sum_{i=2,T} y_{t-1}^2$, of β consistent? You may confine attention to the case where $\beta = 0$.

(b) Suppose that $\theta \neq 0$. Find the probability limit of b in the case where $\beta = 0$, and use this to conclude that the OLS estimator is inconsistent in this case. What is the intuition behind this result?

(c) In order to get around the inconsistency in part (b), a researcher proposes using y_{t-2} as an instrumental variable to obtain the estimator

$$b^{IV} = \sum_{i=3,T} y_t y_{t-2} / \sum_{i=3,T} y_{t-1} y_{t-2}.$$

What is a justification for this suggestion? Find the probability limit of this estimator when $\theta \neq 0$. You may confine attention to the case where $\beta = 0$.

4. Consider the Seemingly Unrelated Regressions model

$$y_1 = X_1\beta_1 + u_1$$

$$y_2 = X_2\beta_2 + u_2$$

in which X_1, X_2 are $n \times k$ non-stochastic matrices with $X_1'X_2 = 0$, $X_1'X_1 = X_2'X_2 = I_k$, and $E[u_j] = 0$, $j = 1,2$ $E[u_j u_h'] = \sigma_{jh} I_n$ for j, h with $\sigma_{jh} = 1$ for $j = h$ and $\sigma_{jh} = r$ for $j \neq h$ (with $-1 < r < 1$), $j, h = 1,2$. Assume r is known.

(a) Find the covariance matrix of the GLS estimator of β_1.

(b) If you knew that $\beta_1 = \beta_2$, show how you can make use of this information in constructing a restricted GLS estimator of β_1 (and of β_2). Compare the covariance matrix of the estimator you propose with that of the unrestricted GLS estimator of β_1 in part (a).

(c) Someone notes that when it is known that $\beta_1 = \beta_2$, we can write, for any choice of the scalar λ,

$$y_1 + \lambda y_2 = (X_1 + \lambda X_2)\beta + u_1 + \lambda u_2,$$

or

$$y^* = X^*\beta + u^*$$

where $y^* = y_1 + \lambda y_2$, $X^* = X_1 + \lambda X_2$, $u^* = u_1 + \lambda u_2$. Find the covariance matrix of u^* and hence the covariance matrix of the OLS estimator of β in the regression of y^* on X^*. Suppose that λ is chosen so that $\lambda = 1$ if $r \leq 0$ and $\lambda = -1$ if $r > 0$: compare the covariance matrix of this estimator with the covariance matrix of the restricted GLS estimator in part (b).