4 Suppose that $X_1, X_2, \ldots, X_n$ are $n = 10$ independent Bernoulli random variables with the property that

$$\mathbb{P}(X_i = 1) = \frac{1}{\beta + 1}, \text{ for } i \in \{1, 2, \ldots, n\},$$

where $0 < \beta < \infty$. The value of β is unknown. For estimating the value of β, we organize an experiment in which we obtain $X_i = x_i$ for $i \in \{1, 2, \ldots, n\}$, as follows:

$$x_1 = x_2 = 1, \qquad x_7 = 1,$$

$$x_3 = x_4 = x_5 = x_6 = 0, \qquad x_8 = x_9 = x_{10} = 0.$$

Note that $n_1 = 3$ of the values $x_1, x_2, \ldots, x_n$ are ones and the other $n - n_1 = 7$ are zeros.

Answer the following questions:

a (5 marks) Write down the log-likelihood function for this problem,

$$\log L(\beta; x_1, x_2, \ldots, x_n).$$

Remember to state the range of values of β for which the log-likelihood is defined. Show your working.

b (4 marks) Find the maximum likelihood estimate of β. In your answer, you should differentiate the log-likelihood with respect to the parameter β, and set to 0 for the maximum. Quote your numerical answer to 4 decimal places.

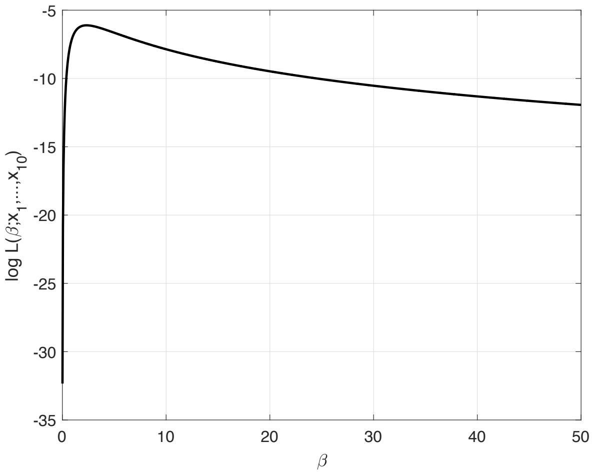

Figure 4: Plot for question 4, part (b).

You should make reference to the graph of the log-likelihood function, which is shown in Figure 4. Show all your working in your answer.

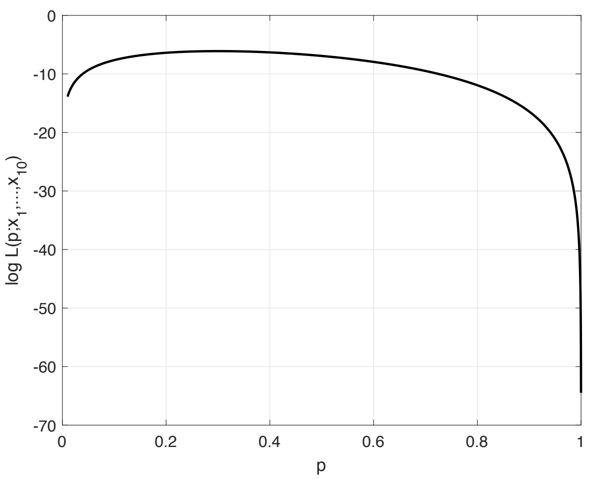

Figure 5: Plot for question 4, part (d).

<u>We define</u>:

$$p = \frac{1}{\beta + 1}.$$

The fact $0 < \beta < \infty$ leads to $0 < p < 1$.

c (4 marks) If we employ the definition above, then the unknown parameter is p. Assuming that the measurements are the same $x_1, x_2, \ldots, x_n$, show that the log-likelihood function has the expression:

$$\log L(p; x_1, x_2, \ldots, x_n) = n_1 \log(p) + (n - n_1) \log(1 - p).$$

Remember to state the range of values of p for which the log-likelihood is defined. Show your working.

d (4 marks) Find the maximum likelihood estimate of p. In your answer, you should differentiate the log-likelihood with respect to the parameter p, and set to 0 for the maximum. Quote your numerical answer to 1 decimal place.

You should make reference to the graph of the log-likelihood function, which is shown in Figure 5. Show all your working in your answer.

e (1 mark) Use the result obtained in part (d) in order to write down the expression of the maximum likelihood estima<u>tor</u> $\widehat{p}$.

f (1 mark) Use the value of $\widehat{p}$ computed in part (d) for finding $\widehat{\beta}$ (see again the definition of p). Compare the value of $\widehat{\beta}$ with the one obtained in part (b).

Total: 55 marks.