Homework Problems for Chapter 10

1. Linear Shooting Method for a Two-point Boundary Value Problem

(A) Consider the differential equation

$$y'' = y' + 2y + \cos(x), \qquad \text{for} \quad 0 \le x \le \frac{\pi}{2},$$

with boundary conditions

$$y(0) = -0.3, \qquad y\left(\frac{\pi}{2}\right) = -0.1.$$

Show that the exact solution is

$$y(x) = -(\sin(x) + 3\cos(x))/10.$$

Implement the shooting method for this problem in Matlab. Use Matlab solver `ode45`, with your choice of error tolerance. You can check your answer by comparing it with the exact solution. Plot your solution, and also the error.

(B) Consider the two-point boundary value problem for the unknown $u(x)$

$$-u'' + 3u = x(1-x), \qquad u(0) = 0, \quad u(1) + u'(1) = 1.$$

Explain in detail how to solve this problem with the shooting method.

2. Non-linear Shooting Method for a Two-point Boundary Value Problem

Consider the differential equation

$$y'' = -(y')^2 - y + \ln(x), \qquad 1 \le x \le 2$$

with the boundary conditions

$$y(1) = 0, \qquad y(2) = \ln 2.$$

Show that the exact solutions is

$$y(x) = \ln x.$$

Implement the shooting method for this problem in Matlab. Use Matlab solver `ode45`. Note that this is a non-linear problem, so you need to use a secant iteration. Since the secant iteration converges quickly if the initial guess is good, it is crucial to get a good initial guess. Try the values $z_1 = 1, z_2 = 0.5$.

You may choose the tolerance to be 10^{-9}, and maximum number of iterations for the secant method to be 5. Plot the approximate solutions together with the exact solution. Plot also the error.

3. Finite Difference Method in 1D

Consider the same equation as in Problem 1A. We will now compute approximate solutions with the finite difference method.

(a). Consider a uniform grid with $h = (b - a)/N$. Set up the finite difference method for the problem. Write out this tri-diagonal system of linear equations for y_i.

(b). Write a Matlab program that computes the approximate solution y_i. You may either use the Matlab solver to solve the linear system, or use the code for tri-diagonal systems (you should find it in a previous homework). Test your program for $N = 10$ and $N = 20$. Plot the approximate solutions together with the exact solution. Plot also the errors.