MATHEMATICS 3000

FINAL PROJECT - 30 POINTS

Problems 1-2-3-4-5-6 are about Differential Functions
Problems 7-8 are about Continuous Functions
Problems 9-10 are about Series-Sequences

Due by 9:00am - 11:30am, Wednesday, April 14, 2021

Name	**M.U.N. Number**

1. [**3 points**] Outline a proof, beginning with basic properties of the real numbers, of the following theorem - if $f : [a, b] \to (-\infty, \infty)$ is a continuous function such that $f'(x) = 0$ for all $x \in (a, b)$, then $f(a) = f(b)$.

2. [**3 points**] Suppose that f is a positive differentiable function on $(0, \infty)$. Prove that: (i) if f' is continuous with $f' \leq 0$ and f'' is bounded, then $\lim_{x\to\infty} f'(x) = 0$; (ii) $\lim_{\epsilon\to 0}\left(\frac{f(x+\epsilon x)}{f(x)}\right)^{\frac{1}{\epsilon}}$ exists finitely and is nonzero for each $x \in (0, \infty)$.

3. [**3 points**] Let $f : [0, 1] \to (-\infty, \infty)$ be continuously differentiable and obey the ordinary differential equation $f''(t) = e^t f(t)$. Show that: (i) if $0 < t_0 < 1$ then f cannot have a positive local maximum at t_0; (ii) if $0 < t_0 < 1$ then f cannot have a negative local minimum at t_0; (iii) if $f(0) = f(1) = 0$ then $f(t) = 0$ for all $t \in [0, 1]$.

4. [**3 points**] Give an example of a function $f : (-\infty, \infty) \to (-\infty, \infty)$ having all three of the following properties: (i) $f(x) = 0$ for $x < 0$ and $x > 2$; (ii) $f'(1) = 1$; (iii) f has derivatives of all orders.

5. [**3 points**] Show that if n is a natural number and α, β are real numbers with $\beta > 0$ then there exists a real function f with derivatives of all orders such that: (i) $|f^{(k)}(x)| \leq \beta$ for $k \in \{0, 1, ..., n-1\}$ and $x \in (-\infty, \infty)$; (ii) $f^{(k)}(0) = 0$ for $k \in \{0, 1, ..., n-1\}$; (iii) $f^{(n)}(0) = \alpha$.

6. [**3 points**] Suppose that $y = f(x) : (-\infty, \infty) \to (-\infty, \infty)$ is infinitely differentiable and has a local minimum at 0. Prove that there exists a disc centered on the y axis which lies above the graph of f and touches the graph at the point $(0, f(0))$.

7. [**3 points**] Suppose that f is continuous function on $(-\infty, \infty)$ which is periodic with period 1, i.e., $f(t + 1) = f(t)$ for all $t \in (-\infty, \infty)$. Prove that: (i) f is bounded and achieves its maximum and minimum; (ii) f is uniformly continuous on $(-\infty, \infty)$; (iii) there is a $t_0 \in (-\infty, \infty)$ such that $f(t_0 + \pi) = f(t_0)$.

8. [**3 points**] A function $f : [0, 1] \to (-\infty, \infty)$ is said to be upper semi-continuous if given $x_0 \in [0, 1]$ and $\epsilon > 0$, there is a $\delta > 0$ such that if $|x - x_0| < \delta$, then $f(x) < f(x_0) + \epsilon$. Show that an upper semi-continuous function f on $[0, 1]$ is bounded above and attains its maximum value at some point $x_* \in [0, 1]$, and answer the question of whether or not there is a corresponding result on the so-called 'lower semi-continuous function'.

9. [**3 points**] Suppose that (a_j) is a sequence of positive numbers. Show that: (i) $\sum_{j=1}^{\infty} a_j < \infty \Rightarrow \sum_{j=1}^{\infty} \sqrt{a_j a_{j+1}} < \infty$; (ii) $\sum_{j=1}^{\infty} \sqrt{a_j a_{j+1}} < \infty \not\Rightarrow \sum_{j=1}^{\infty} a_j < \infty$; (iii) if $\sum_{j=1}^{\infty} a_j < \infty$ then there is a sequence of positive numbers (c_j) such that $\lim_{j\to\infty} c_j = 0$ and $\sum_{j=1}^{\infty} c_j a_j < \infty$.

10. [**3 points**] Evaluate two limits: (i) $\lim_{n\to\infty} \cos\frac{\pi}{2^2}\cos\frac{\pi}{2^3}\cdots\cos\frac{\pi}{2^n}$; (ii) $\lim_{n\to\infty}(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n})$.