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4. **(6 marks)** Consider the regression model $\mathbf{y} = \mathbf{X}\beta + \mathbf{u}$, where the Gauss-Markov assumptions hold. Let $\hat{\beta}$ be the OLS estimator of $\beta$, $\mathbf{Z} = \mathbf{G}(\mathbf{X})$ be an $n \times (k+1)$ matrix function of $\mathbf{X}$ and $\mathbf{Z}'\mathbf{X}$ be a $(k+1) \times (k+1)$ non-singular matrix.

   a. Prove that the estimator $\tilde{\beta} = (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'\mathbf{y}$ is unbiased.

   b. State $\text{var}(\tilde{\beta})$ as a function of $\mathbf{Z}$, $\mathbf{X}$ and $\sigma^2$.

   c. ***Briefly*** explain which estimator is better: $\hat{\beta}$ or $\tilde{\beta}$.

5. **(6 marks)** Consider the regression model $y = \beta_0 + \delta_0 d + \beta_1 x + \delta_1 (d \cdot x) + u$, where $d$ is a dummy variable and $x$ is a quantitative variable. For simplicity, assume that $u = 0$.

   a. Calculate the unique (strictly positive) value of $x$ for which $\hat{y}\big|_{d=0} = \hat{y}\big|_{d=1}$.

   b. State the necessary parameter restriction(s) for part (a) to be true.

   c. State a test statistic that tests whether there is a structural difference between the two cases represented by the dummy variable.

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6. **(10 marks)** Consider the regression model $\mathbf{y} = \mathbf{X}\beta + \mathbf{u}$, where $\mathbf{u}\mathbf{u}' = \mathbf{V} = \begin{bmatrix} \sigma_1^2 \mathbf{I}_{n_1} & 0 \\ 0 & \sigma_2^2 \mathbf{I}_{n_2} \end{bmatrix}$ and $n_1 + n_2 = n$. Heteroscedasticity will not cause the OLS estimators to be biased or inconsistent, but it will make problems for standard errors, *t*-statistics and various significance tests. To fix it, a "transformation matrix", $\mathbf{P}$, is needed, such that the transformed error vector, $\tilde{\mathbf{u}} \equiv \mathbf{P}\mathbf{u}$, satisfies $\tilde{\mathbf{u}}\tilde{\mathbf{u}}' = \sigma^2 \mathbf{I}$.

a. Derive a diagonal matrix, $\mathbf{P}$, that satisfies $\mathbf{P}\mathbf{P}' = \mathbf{V}^{-1}$.

b. Restate the model using the transformation matrix.

c. State the OLS estimator, $\hat{\beta}$, for the transformed model in terms of $\mathbf{X}$, $\mathbf{y}$ and $\mathbf{V}$.

d. Prove that $\hat{\beta}$ is unbiased.

e. State the variance-covariance matrix in terms of $\mathbf{X}$ and $\mathbf{V}$.

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7. **(6 marks)** Consider the regression model $y = \beta_0 + \beta_1 x + u$, where $x$ is unobservable. For each $y_i$, there are $m$ measures on $x$: $z_h = x + e_h$ for $h = 1,\ldots,m$. Assume that $x$ is uncorrelated with $u$, $e_1$, … and $e_m$ and that the measurement errors are pairwise uncorrelated with the same variance, $\sigma_e^2 > 0$. Let $w = \frac{1}{m}(z_1 + \ldots + z_m)$ be the average of the measures on $x$ so that, for each observation $i$, $w_i = \frac{1}{m}(z_{i1} + \ldots + z_{im})$. Let $\bar{\beta}_1$ be the OLS estimator from the regression of $y$ on a constant and $w$, using a random sample of data.

a. Prove that $\bar{\beta}_1 = \dfrac{\beta_1 \sigma_x^2}{\sigma_x^2 + \frac{\sigma_e^2}{m}}$.

b. Prove that $\bar{\beta}_1$ is a biased estimator.

c. ***<u>Briefly</u>*** explain the sign of the bias and why it disappears as $m$ increases.

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8. **(4 marks)** Consider the regression $y_t = \alpha_0 + \delta_0 x_t + \delta_1 x_{t-1} + \delta_2 x_{t-2} + u_t$. If $x_t$ increases one unit, it will have a long-run effect on *y* equal to $\theta \equiv \delta_0 + \delta_1 + \delta_2$. A limitation of this model is that the standard error of this long-run effect cannot be obtained easily.

   a. Eliminate $\delta_0$ from the model and isolate $\theta$.

   b. To ensure that $\theta$ has economic meaning, briefly explain the relationship between $\Delta x_t$, $\Delta x_{t-1}$ and $\Delta x_{t-2}$.

9. **(4 marks)** Let $x_t = z + e_t$, where $z \sim iid\, N\left(0, \sigma_z^2\right)$ does not change over time, $e \sim iid\, N\left(0, \sigma_e^2\right)$ and *z* is uncorrelated with *e*.

   a. Prove that $x_t$ is covariance stationary. (Hint: covariance stationary means that $E\left(x_t\right)$ is constant, $\mathrm{var}\left(x_t\right)$ is constant and $\mathrm{cov}\left(x_t, x_{t+h}\right)$ is independent of *t*.)

   b. Calculate $\mathrm{cor}\left(x_t, x_{t+h}\right)$ for all $h > 0$.