**MATH 447/647: Probability Models**
**Fall 2020 Semester**
**Homework 8: Due Monday December 14, 5:30pm ET.**

## Instructions

- Write your full name, "Homework 8", and the date at the top of the first page.
- Show all work, including each step of your solution, to earn maximal partial credit.
- Each question has multiple parts. Write legibly and neatly. Box your final answers.
- Use Genius Scan or a similar application to convert your solutions to .pdf format.
- Submit a single .pdf file to Gradescope under the assignment "Homework 8".
- If you have any questions, email me or come to office hours (WF 11:00am-12:00pm)
- You are encouraged to work together (on Piazza) but must write up your own solutions.

**Assignment** (4 Problems: 25 + 25 +25 + 25 = 100 points total.)

☐ **Problem 1** Let $X(t)$ be the birth-death process defined as a continuous time Markov chain in Example 1 in Lecture 21 with state space $\mathbb{X} = \{a, b\}$ and 1-step transition rates $\lambda = q_{ab} = 5$ and $\mu = q_{ba} = 1$.

- 1.1 [10 points] Given that $X(0) = a$, what is the probability that the chain stays in the state $a$ for all times $t$ in the time interval $[0, 0.3]$?
- 1.2 [15 points] Given that $X(0) = a$, what is the probability that the chain is in the state $b$ at time $t = 0.8$ and that also the chain has only jumped from $a$ to $b$ **once** in the time interval $[0, 0.8]$?

☐ **Problem 2** Let $X(t)$ be the continuous time Markov chain defined in Example 2 in Lecture 21 with state space $\mathbb{X} = \{a, b, c\}$ and infinitesimal generator

$$\mathbf{Q} = \begin{bmatrix} -6 & 3 & 3 \\ 4 & -12 & 8 \\ 15 & 3 & -18 \end{bmatrix}$$

which encodes the 1-step transition rates $q_{ij}$ and decay rates $v_i$ by $q_{ii} = -v_i = -\sum_{j \neq i} q_{ij}$.

- 2.1 [10 points] Determine the $3 \times 3$ matrix $\mathbf{P}$ with entries

$$\mathbf{P}_{ij} = \begin{cases} \frac{q_{ij}}{\sum_{k \neq i} q_{ik}} & \text{if} \quad i \neq j \\ 0 & \text{if} \quad i = j. \end{cases}$$

- 2.2 [15 points] Given that $X(0) = b$, what is the probability that this continuous time Markov chain visits the state $a$ before visiting $c$? Equivalently, what is the probability that this process started at $b$ makes its first jump from $b$ to $a$ (and not from $b$ to $c$)?

☐ **Problem 3** Let $X(t)$ be the birth-death process from Example 1 in Lecture 22 with state space $\mathbb{X} = \{0, 1\}$ and infinitesimal generator

$$\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

where $\lambda, \mu > 0$ are two positive constants. This model generalizes the chain in Problem 1. In Lecture 22, we used Kolmogorov's backward equations to derive the formula

$$P_{00}(t) = P(X(t) = 0|X(0) = 0) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

for the transition probabilities $P_{00}(t)$ at any time $t \geq 0$. Kolmogorov's insight was that in order to solve for the unknown $P_{00}(t)$, one has to temporarily enlarge the number of unknown transition probabilities one cares about – specifically, to $P_{00}(t)$ *and* $P_{10}(t)$ – in order to solve for both unknowns at once and hence solve for $P_{00}(t)$.

- 3.1 [5 points] For any $\lambda, \mu$, what is the transition probability $P_{00}(0)$ at $t = 0$?
- 3.2 [10 points] If $\mu = 0$ but $\lambda > 0$, what is the transition probability $P_{00}(t)$?
- 3.3 [10 points] If $\mu = \lambda > 0$, what is the limiting transition probability $\lim_{t\to\infty} P_{00}(t)$?

☐ **Problem 4** In the context of Problem 3, in Lecture 22 we derived a formula for $P_{00}(t)$ by studying two unknown transition probabilities $P_{00}(t)$ and $P_{10}(t)$. For birth-death processes with more than two states, we also learned that one has to consider more than two unknown transition probabilities. Consider the $M/M/1$ queue, the birth-death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = \cdots = \lambda$$

$$\mu_1 = \mu_2 = \mu_3 = \cdots = \mu$$

for birth and death rates $\lambda, \mu > 0$.

- 4.1 [15 points] State the system Kolmogorov backward equations for the transition probabilities of the $M/M/1$ queue. You do **not** have to solve this system.
- 4.2 [10 points] Simplify your answer in the case $\mu = 0$.

☐ **Bonus** [X points] For any birth-death process with birth and death rates $\lambda_n, \mu_n > 0$ at the state $n$ in $\mathbb{X} = \mathbb{N}$, consider the quantity $\pi_n$ defined by the formula

$$\pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n \left(1 + \sum_{k=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k}\right)}.$$

Evaluate $\pi_n$ in the case of the birth-death process with rates

$$\lambda_n = Ln$$

$$\mu_n = Mn$$

for some $M, L > 0$ satisfying $L < M$.