Homework Problems for Chapter 7

1. Comparing various methods

Solve the linear system with the four methods below (Do not use Matlab).

$$\begin{cases} 4\,x_1 + 3\,x_2 = 24 \\ 3\,x_1 + 4\,x_2 - x_3 = 30 \\ - x_2 + 4\,x_3 = -24 \end{cases}$$

(a). Gaussian elimination (tridiagonal system).

(b). Jacobi's method.

(c). Gauss-Seidel's method.

(d). The SOR-method, with $\omega = 1.25$.

For the methods in (b), (c) and (d), write out the general iteration scheme, then do 2 iterations for each method, with initial guess

$$x^0 = [24/4,\ 30/4,\ -24/4] = [6,\ 7.5,\ -6].$$

Among the methods (b), (c), (d), which seems to work better for this example? Please comment.

2. SOR in Matlab

(a). Write a Matlab function which solves a system of linear equations $Ax = b$, with successive over relaxation (SOR) iterations. Assume here that A is a banded matrix with band width d, (so that $a_{ij} = 0$ for $|i-j| > d$). The inputs of the function are: A, b, a starting vector x_0, the band-width d, the relaxation parameter w, an error tolerance ε and the maximum number of iterations. The iteration stops when the error (you may use the residual $r = Ax - b$ measured in certain norm) is less than the tolerance, or when the maximum number of iterations is reached. The function should return the solution vector x and the number of iterations.

The first few lines in the function should look like this:

```
function [x,nit]=sor(A,b,x0,w,d,tol,nmax)
% SOR : solve linear system with SOR iteration
% Usage: [x,nit]=sor(A,b,x0,omega,d,tol,nmax)
% Inputs:
%       A : an n x n-matrix,
%       b : the rhs vector, with length n
%       x0 : the start vector for the iteration
```

```
%       tol: error tolerance
%       w: relaxation parameter, (1 < w < 2),
%       d : band width of A.
% Outputs::
%       x : the solution vector
%       nit: number of iterations
```

(b). Use your function `sor` to solve the following tridiagonal system, with

$$A = \begin{pmatrix} -2.011 & 1 & & & & & & & \\ 1 & -2.012 & 1 & & & & & & \\ & 1 & -2.013 & 1 & & & & & \\ & & 1 & -2.014 & 1 & & & & \\ & & & 1 & -2.015 & 1 & & & \\ & & & & 1 & -2.016 & 1 & & \\ & & & & & 1 & -2.017 & 1 & \\ & & & & & & 1 & -2.018 & 1 \\ & & & & & & & 1 & -2.019 \end{pmatrix},$$

$$b = \begin{pmatrix} -0.994974 \\ 1.57407 \cdot 10^{-3} \\ -8.96677 \cdot 10^{-4} \\ -2.71137 \cdot 10^{-3} \\ -4.07407 \cdot 10^{-3} \\ -5.11719 \cdot 10^{-3} \\ -5.92917 \cdot 10^{-3} \\ -6.57065 \cdot 10^{-3} \\ -0.507084 \end{pmatrix}, \qquad x^0 = \begin{pmatrix} 0.95 \\ 0.9 \\ 0.85 \\ 0.8 \\ 0.75 \\ 0.7 \\ 0.65 \\ 0.6 \\ 0.55 \end{pmatrix}.$$

Here b is the load vector and x^0 is the initial guess. Solve the system with an error tolerance $\varepsilon = 10^{-4}$, setting 100 as the maximum number of iterations. Try different values of w between 1 and 2 (this is called over-relaxation), for example $w = 1.0, 1.1, 1.2, \cdots, 1.9$. Search for the value of w that gives the fastest convergence (requiring the smallest number of iterations). Make a plot of number of iterations as a function of w.

What to hand in? Your Matlab file `sor.m`, the script, the running result, the plot, and whatever comments you have.

3. Jacobi iterations in Matlab

Write a Matlab function, using Jacobi's method to solve a system of linear equations. Do it similarly as in Problem 2(a), but only for tri-diagonal systems.

Test it on the same system in Problem 2b), with error tolerance $\varepsilon = 10^{-4}$ and setting the maximum number of iterations =100. Compare the result with the result from SOR. Which method converges faster? Put your comments.

What to hand in? Your Matlab file `jacobi.m`, the script, the running result, and whatever comments you have.

4. More practice on various methods

Do not use Matlab for this problem.

Consider the system of linear equations

$$\begin{aligned} 5x + 4y - 2z &= 2 \\ -2x + 8y - 3z &= 6 \\ x + y - 7z &= 5 \end{aligned}$$

- Perform one step Jacobi iteration, using $x_0 = y_0 = z_0 = 1$ as starting value.
- Perform one step Gauss-Seidel iteration, using $x_0 = y_0 = z_0 = 1$ as starting value.
- Do Jacobi iterations converge for this system? Do Gauss-Seidel iterations converge for this system? Why?