Discussion of the Pipe Problem

This problem concerns the flow of oil through a pipe that is half buried in the ground. This pipe configuration results in uneven heating such that the temperature varies on the circumference of the pipe causing a steady temperature distribution (due to conduction) throughout the oil in the pipe. This variation in temperature, in turn, causes a variation in the viscosity of the oil throughout the pipe cross-section. Because the viscosity varies, the axial velocity distribution is no longer the simple Hagen-Poiseuille pipe flow distribution, but instead is substantially different. Finally, if the velocity distribution is different, how does this change the volumetric flow rate of oil in the pipe? This is important to know accurately if you are delivering oil through this pipe to a customer; the financial return is based on how much oil is actually delivered.

One way to estimate the volumetric flow rate using a *back of the envelope* calculation is to assume Hagen-Poiseuille pipe flow. For this simple pipe flow, the flow rate is available analytically. For the viscosity needed in this simple formula, one can use a viscosity for oil at a temperature averaged over the pipe cross-section. How good is this estimate?

The more accurate way to determine the flow rate is to compute, using CFD, the temperature field, the viscosity field and then the velocity field in this flow. Once the velocity field has been determined, one can compute the flow rate numerically by integrating over the pipe cross-section. This more accurate answer can then be compared to the simple estimate described above. This is the approach that will be taken for this part of the Course Project. You will answer the question posed above by comparing the answer given by the simple estimate to the more precise answer determined through the application of CFD techniques. Now, we turn to the construction of the grid.

Grid Development

The circular geometry of the pipe suggests the use of cylindrical coordinates. Because we are not varying any quantities along the axis of the pipe, our problem is a 2-D problem. As we consider constructing a grid, two issues become important. Is there any symmetry in the problem? If the solution for the temperature and velocity fields is symmetric, then we do not have to compute the solution over the entire domain. Instead, we can save on computational resources and compute the solution for half of the domain. Then, it is a simple matter to take the computed solution and reflect it across the symmetry boundary to get the solution for the rest of the domain. As we consider the temperature and velocity fields below, we will see that the solution is symmetric about a vertical diameter of the pipe.

> ***Consideration of any symmetry that may exist in a problem can greatly reduce the computational resources that are required.***

The second issue concerns the implementation of the code that generates the grid. When approaching a problem for the first time, it is not clear how ***dense*** the grid should be, that is, the number of radial and azimuthal spacings that will be required. Therefore, one should develop equations to create the grid that use the number of spacings as a variable. Then, this variable can be changed one place in the code and the grid will update accordingly. The equally-spaced polar grid that we will develop is shown in Fig. 2.7.

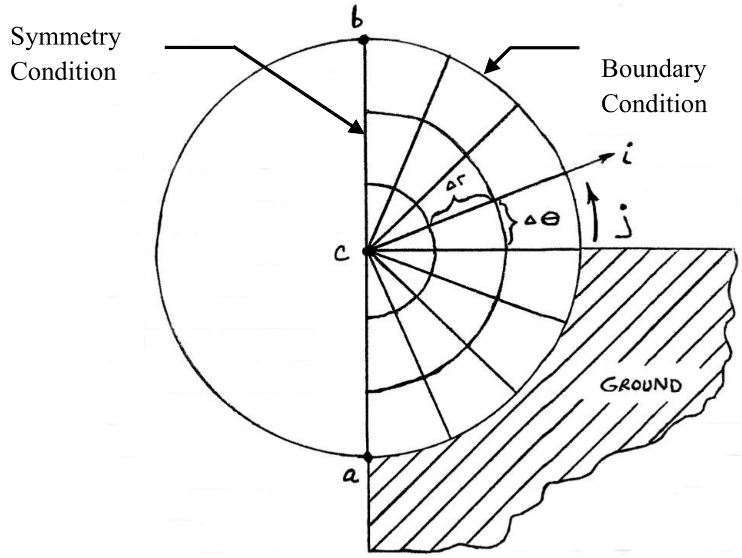

Fig. 2.7 Grid for the pipe problem.
Shown with $N = 4$ in r-direction and with $N = 9$ in the θ-direction for convenience.

As the figure shows we will compute the solution for half of the domain. Let the number of grid lines in the radial and azimuthal directions be given by N. Therefore, the number of intervals (spacings) in these two directions will be $N - 1$. The radial and azimuthal spacings are then given by

$$\Delta r = \frac{R}{N-1} \text{ and } \Delta\theta = \frac{\pi}{N-1} \ . \tag{2.27}$$

The radial and azimuthal locations of the i,j^{th} grid point are determined from

$$r_i = (i-1)\Delta r \text{ and } \theta_j = -\frac{\pi}{2} + (j-1)\Delta\theta \ . \tag{2.28}$$

for $i, j = 1, \cdots, N$. Note that we are using a grid that is equally spaced and with the same number of intervals in both directions, whereas, Fig. 2.7 displays a different number of intervals in each direction for ease of viewing. For implementation, it will be useful for $N - 1$ to be a multiple of 16 such as 64, 128, 256, 512, etc, which means that N will have values such as 65, 129, 257, 513, etc. (You are only required to use $N = 65$, but variation of this quantity to see its effect is a simple matter).

We will find it useful, for a given grid size, to be able to determine the value of j that corresponds to the ray: $\theta = 0$. If we call this value $j = j_{mid}$, then it is given as

$$j_{mid} = \frac{N-1}{2} + 1 \ . \tag{2.29}$$

As can be seen from Fig. 2.7, when $N = 9$, then $j_{mid} = 5$.

We will also find it useful to be able to represent our polar grid as a square mesh. This is helpful when visualizing two-dimensional arrays of values. The square grid is shown in Fig. 2.8. Points a, b and c are provided in both figures as an aid to locating these points. Note that the center point of the polar grid is repeated multiple times in the square grid, as it will be in the 2-D array that stores the dependent variable.

Next, we will consider the solution of the temperature field for this problem.

Temperature Field

The temperature boundary condition on the pipe wall from the project description is given as

$$T_s = 125°F + 50°F \sin\theta \quad . \tag{2.30}$$

To develop intuition about this problem, ask yourself the following series of questions. Where is the temperature on the pipe wall the highest? Where is it the lowest? What is the temperature on

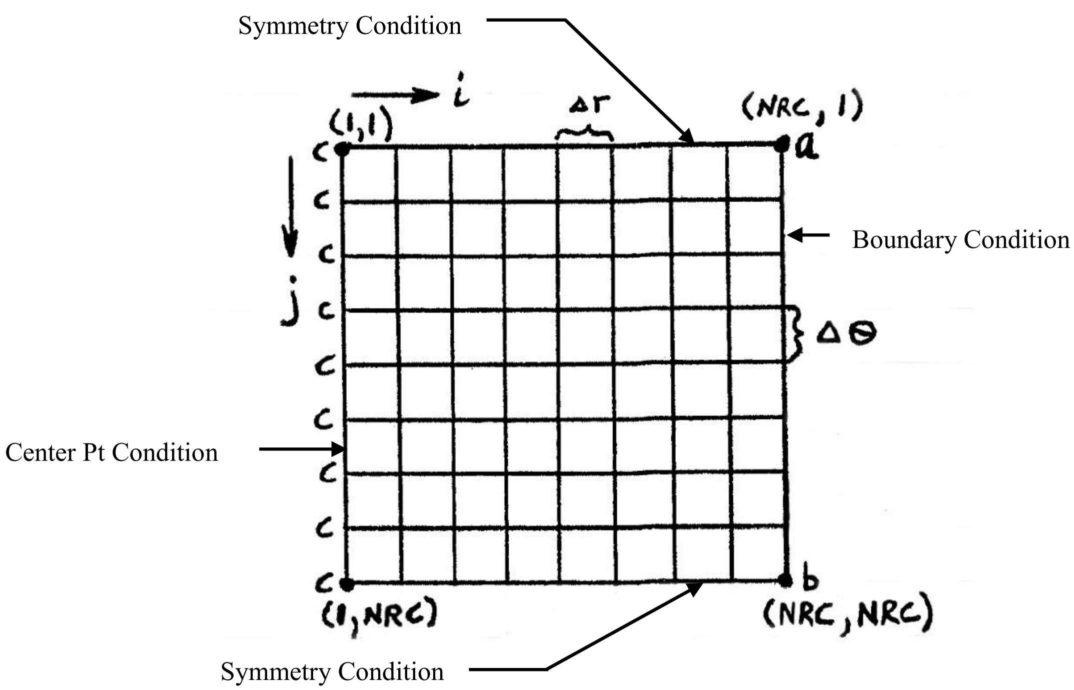

Fig. 2.8 Rectangular grid representation for the pipe problem.
$N = NRC$ in this figure.

the pipe wall at $\theta = 0$ and at $\theta = \pi$? Are they the same? If there is symmetry about the vertical diameter, then what do you think the temperature might be at every point along the horizontal diameter? Therefore, what would be the temperature at the center of the pipe? If the temperatures are higher above the horizontal diameter and lower below the horizontal diameter, then what value could be used to represent the average temperature in the pipe?

The project description gives the governing equation for the temperature field $T(r,\theta)$ as

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \quad . \tag{2.31}$$

This is the steady heat conduction equation written in cylindrical coordinates; it can also be written as

$$\nabla^2 T = 0 \ , \tag{2.32}$$

because the ∇^2 operator in cylindrical coordinates is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} = 0 \ , \tag{2.33}$$

and we are not considering any variation in the axial direction. What kind of equation is Eq. 2.32?

So, we proceed as we did for the first problem in the project. Discretize Eq. 2.31 by inserting second order accurate central differences for the derivatives $\frac{\partial^2 T}{\partial r^2}, \frac{\partial T}{\partial r}$ and $\frac{\partial^2 T}{\partial \theta^2}$. For example,

$$\left[\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta r^2}\right] + \frac{1}{r_i}[-] + \frac{1}{r_i^2}[-] = 0 \ . \tag{2.34}$$

Then, solve the resulting equation for $T_{i,j}$ in order to get an equation that can be iterated. Get an equation that looks like

$$T_{i,j} = C_i[Term\ 1 + Term\ 2 + Term\ 3] \ , \tag{2.35}$$

where C_i is a collection of known values including r_i, which is the reason for the subscript. This is the equation that must be iterated over all interior points $i, j = 2, \cdots, N-1$. Now, let's consider the various boundaries in the problem.

Along the $\theta = -\frac{\pi}{2}$ radius ($j = 1$), what is the symmetry condition? The condition is that the azimuthal component of the heat flux vector is zero. Namely,

$$\begin{gathered} \underline{q} = -k\nabla T \\ q_\theta = -\frac{k}{r}\frac{\partial T}{\partial \theta} = 0 \ \ \theta\text{-component} \\ \frac{\partial T}{\partial \theta} = 0 \end{gathered} \quad . \tag{2.36}$$

The top equation indicates that the heat flux vector is given by the negative of the thermal conductivity times the gradient of temperature. In other words, ***heat flows down the temperature gradient***. The middle equation is the azimuthal component of the top equation written in cylindrical coordinates. The middle equation reduces to the bottom equation. The reason that no heat flows across the symmetry boundary is that there is no temperature gradient across the boundary; this means that the temperature on one side of the symmetry boundary is the same as the temperature at the corresponding grid point on the other side of the boundary. To see this, insert a central difference for the derivative $\frac{\partial T}{\partial \theta}$ into the bottom equation of Eq. 2.36 and simplify the resulting algebraic equation. This will be the ***symmetry condition***.

Now, Eq. 2.35 will have terms containing $T_{i,j-1}$. For interior points, these terms will always exist; however, along the symmetry boundary $j = 1$, this means that we need grid points $T_{i,0}$,

which do not exist. What do we do? The answer is that we use the symmetry condition to replace terms containing $T_{i,j-1}$ with terms that we *can evaluate* along $j = 1$. Therefore, by invoking the symmetry condition, we will use a simplified form of Eq. 2.35, which *can only be used to iterate points along the symmetry boundary* $j = 1$. So, just like interior points, symmetry points must also be iterated.

Similarly, along the symmetry boundary $j = N$, we will have a problem with terms containing $T_{i,j+1}$ that will be present in Eq. 2.35. So, we use the symmetry condition again to replace terms containing $T_{i,j+1}$ with terms that we *can evaluate* along $j = N$. Therefore, by invoking the symmetry condition, we will use a *different* simplified form of Eq. 2.35, which *can only be used to iterate points along the symmetry boundary* $j = N$.

What do we do on the boundary $i = N$? This corresponds to the wall of the pipe, and we have a ***boundary condition*** along this wall given by Eq. 2.30. We do not iterate boundary points. We simply set the value of the temperature at each boundary point, and maintain these values throughout the computation. Note that the boundary condition is not zero (as it was for the Laplace Equation in the first problem), and we will therefore have a nonzero temperature solution.

This leaves one final boundary as can be seen in Fig. 2.8. What equation should be used to iterate the center point? We know two facts about the center point. The first is

$$T_{1,j+1} = T_{1,j-1} = T_{1,j} \quad , \tag{2.37}$$

for $j = 1, \cdots, N$. This simply states that all of the center points shown in Fig. 2.8 are *the same point*. The second fact is that along the ray $\theta = 0$ corresponding to $j = j_{mid}$, we know that

$$T_{i+1,j} = T_{i-1,j} \quad , \tag{2.38}$$

for $j = j_{mid}$. Plug Eqs. 2.37 and 2.38 into Eq. 2.35 to get (after some algebra) *a very simple equation*. This equation can only be used to iterate the single point at $i = 1$ and $j = j_{mid}$. After this point has been updated, then following Eq. 2.37, update all of the other center points to this new value.

Of course, there is one other option for the center point. *We know the temperature* at the center point! So, the other option is to treat the center point like a boundary condition and maintain it throughout the computation.

This center point approach is much easier, but this is ***not*** the best option to take. Here's why. You will also need to solve for the center point velocity when we proceed to consider the velocity solution. We do ***not*** know the velocity at the center of the pipe. Therefore, we will not have the simple approach for the center point velocity. We will have to use the iterated approach. However, there will be no way to check that code in advance. If you iterate for the center point for the temperature solution, the center point temperature ***better be*** the expected center point temperature upon convergence. This will be a check that your code is correct, and you will be better prepared to compute the center point velocity.

What initial temperatures should be specified for all points that are not boundary points prior to iteration? The most sensible choice is to set these points to the ***average temperature*** because we know during iteration that some temperatures will go higher and some lower. What iteration method should be used? We will use Gauss-Seidel as opposed to Jacobi to ensure faster convergence. As before use the L_2 norm and monitor for the condition $\Delta\varepsilon \leq \varepsilon_{min}$. Keep track of the iterations and the elapsed time.

Summarizing, the following steps should be performed (in any order) during each iteration.

1. Iterate the simplified equation on the symmetry boundary at $j = 1$.
2. Iterate the full equation, Eq. 2.35 on interior points.
3. Iterate the other simplified equation on the symmetry boundary at $j = N$.
4. Iterate the very simple equation for the center point.
5. Maintain boundary points at the boundary condition.

For this part of the pipe problem, an exact solution is available for the temperature field! Although not required, the motivated student may wish to solve Eq. 2.31 by using the Separation of Variables technique. Once the exact solution has been obtained, one may code the analytical solution and compare the exact solution to the computed solution (after convergence) at each grid point as a check on the computation. An obvious check would be to save the differences in an array and report the maximum difference as a measure of the goodness of the computed solution.

Viscosity Field

After obtaining a converged temperature field, one has a 2-D array of temperatures $T_{i,j}$ at the grid points. To obtain the viscosity field $\mu(r,\theta)$, no iteration is required. One simply computes a 2-D array of viscosities at the grid points using

$$\mu_{i,j} = \mu_0 \, e^{\frac{c}{T_{i,j}+d}} \quad , \tag{2.39}$$

where μ_0, c and d are constants that are provided.

Velocity Field

We now proceed to compute the velocity field for the axial velocity component $u(r,\theta)$ in the pipe. The governing equation is given as

$$\frac{dp}{dz} = \frac{1}{r}\frac{\partial}{\partial r}\left(\mu\, r\frac{\partial u}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\mu}{r}\frac{\partial u}{\partial \theta}\right) \quad , \tag{2.40}$$

where dp/dz is a constant that is provided. Note in Eq. 2.40 that ***μ is not constant***. The differentiation of a product that appears in two terms of the equation means that when the differentiation is carried out, you will obtain derivatives of μ as well as derivatives of u.

Before proceeding to consider this equation further, ask yourself the following series of questions. What is the boundary condition for the velocity field on the pipe wall (answer this yourself)? Do we have symmetry conditions as we did for the temperature field (yes)? What is the velocity at the center point (unknown in advance)? What velocity values should be given to interior points initially?

To answer this last question, consider that, for a constant temperature distribution, the velocity field would be the Hagen-Poiseuille parabolic profile. Therefore, it makes sense to use this equation to define the initial velocities. Review pp. 274-280 in your text. Note that when the Navier-Stokes equation is given in Eq. 6.9, the viscosity (buried in the kinematic viscosity term $\nu = \mu/\rho$) is already considered as a constant and has been pulled outside of the derivative operations. This is what permits an analytic solution, and the equation for the axial velocity is given in Eq. 6.16. What value for μ should be used in this equation to define the initial velocities? Use a value μ_{avg} found by evaluating Eq. 2.39 at the average temperature in the pipe. Use the constant dp/dz provided in the project description for the pressure gradient required in Eq. 6.16. Also note in your text that the average velocity for a Hagen-Poiseuille parabolic profile can be found in Eq. 6.25. The average velocity times pipe cross-sectional area gives the volumetric flow rate Q. This is the source of the flow rate estimate provided in the project handout as Eq. 6.

We can now proceed to *discretize* Eq. 2.40. First, you must carry out all derivative operations implied in the equation. Then, introduce second order accurate central differences to estimate the derivatives that appear. For example, two of the derivatives that will exist are

$$\frac{\partial \mu}{\partial r} = \frac{\mu_{i+1,j} - \mu_{i-1,j}}{2\Delta r} \text{ and } \frac{\partial u}{\partial r} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta r} \quad . \tag{2.41}$$

There will be other derivative expressions as well; each will require a central difference. For the expressions containing $\mu_{i,j}$, you already have these numbers! You just plug in these values during the iteration. That is, $\mu_{i,j}$ is not iterated, only the dependent variable $u_{i,j}$ is iterated. After inserting the central difference expressions, solve the equation for $u_{i,j}$ to get an equation that can be iterated. Get an equation that looks like

$$u_{i,j} = C_{ij}[Term\ 1 + Term\ 2 + Term\ 3 + Term\ 4 + Term\ 5] \ , \tag{2.42}$$

where C_{ij} is a collection of known values including r_i and $\mu_{i,j}$, which is the reason for the subscripts. This is the equation that must be iterated over all interior points $i, j = 2, \cdots, N-1$.

Then, as with the temperature equation, you will proceed to determine simplified forms of Eq. 2.42 to use on the two symmetry boundaries by substituting for any $u_{i,j-1}$ expressions that may appear on the $j = 1$ boundary using the symmetry condition, and by substituting for any $u_{i,j+1}$ expressions that may appear on the $j = N$ boundary. The same center point conditions given in Eqs. 2.37 and 2.38 apply, using u in place of T. Insert these conditions into Eq. 2.42 and simplify to get a very simple equation (but not the same equation as for the temperature field) that can be iterated at the center point.

We will again use Gauss-Seidel as the iteration method. As before use the L_2 norm and monitor for the condition $\Delta\varepsilon \leq \varepsilon_{min}$. Keep track of the iterations and the elapsed time.

Summarizing, the following steps should be performed (in any order) during each iteration.

1. Iterate the simplified equation on the symmetry boundary at $j = 1$.
2. Iterate the full equation, Eq. 2.42 on interior points.
3. Iterate the other simplified equation on the symmetry boundary at $j = N$.
4. Iterate the very simple equation for the center point.
5. Maintain boundary points at the boundary condition.

After obtaining a converged velocity field, one has a 2-D array of velocities at the grid points.

Volumetric Flow Rate

Now, we get to the main point of the problem. Precisely how much oil is flowing through this pipe? From calculus, recall that the average velocity over a cross-sectional area is obtained from the following integral

$$\overline{u} = \frac{1}{A}\int_A \underline{u} \cdot \underline{dA} \quad . \tag{2.43}$$

The volumetric flow rate is this average velocity times the pipe cross-sectional area, namely

$$Q = \overline{u}A = \int_A \underline{u} \cdot \underline{dA} \quad . \tag{2.44}$$

For the circular pipe, the area integral is determined as

$$Q = \int_0^{2\pi} \int_0^R u\, r\, dr d\theta \quad . \tag{2.45}$$

However, we only have velocities for half of the pipe, so we integrate over half of the pipe and multiply by two using the following form of the equation

$$Q = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^R u\, r\, dr d\theta \quad . \tag{2.46}$$

How do we perform the integration numerically? There are a host of methods that are available for numerical integration. A simple method is the *Trapezoidal Rule*. The trapezoidal rule applied to a function $f(x)$ is

$$\int_{x_1}^{x_N} f(x)dx = \Delta x\left[\frac{1}{2}f(x_1) + f(x_2) + \cdots + f(x_{N-1}) + \frac{1}{2}f(x_N)\right] + O(\Delta x^3) \quad , \tag{2.47}$$

where the coefficients are all one except for the first and last term. An implementation of this method for the double integral in Eq. 2.46 is given as follows.

The inner integral can be found as

$$\begin{gathered} usum_j = \textstyle\sum_2^{N-1} r_i\, u_{i,j} \\ uint_j = \Delta r\left[\frac{1}{2}r_1\, u_{1,j} + usum_j + \frac{1}{2}r_N\, u_{N,j}\right] \quad , \end{gathered} \tag{2.48}$$

where $usum$ and $uint$ are arbitrarily chosen array names. Then, the outer integral is

$$USUM = \sum_2^{N-1} uint_j$$
$$Q = \Delta\theta\left[\frac{1}{2}uint_1 + USUM + \frac{1}{2}uint_N\right] \quad . \qquad (2.49)$$

Finally, we recall that we must double this answer to get the flow rate for the entire pipe

$$Q_{act} = 2Q \quad . \qquad (2.50)$$

This is the actual flow rate in the pipe accounting for the temperature variation, which causes the viscosity, and therefore the velocity, to vary.

The final step is to determine the difference in percent between the actual flowrate Q_{act} and the estimated flowrate using the Hagen-Poiseuille profile Q_{est}. This can be determined from

$$\Delta Q\% = \left|\frac{Q_{act} - Q_{est}}{Q_{act}}\right| \times 100 \quad . \qquad (2.51)$$

A numerical integration technique more accurate than the trapezoidal rule is *Simpson's Rule*. This is the technique used in my implementation; it is only marginally more complicated to program. The motivated student may wish to employ this technique instead.

Pipe Project Products

A report should be prepared discussing the temperature, viscosity and velocity solutions. Graphs of the temperature, viscosity and velocity solutions in the form of contour plots or 3-D surface plots should be provided. At the very least, one should provide the contour plots of the solution for half of the pipe in rectangular form using the grid in Fig. 2.8. These plots are, however, difficult to interpret. A better solution, if Matlab is used, is to generate ***polar contour plots***. One should look up the appropriate commands in Matlab help. To create a polar contour plot for the entire pipe, one must create some code to copy the computed solution (for temperature, viscosity or velocity) to the other half of the domain. The resulting plots are quite appealing; examples for the temperature and velocity solutions are provided in Figs. 2.9 and 2.10.

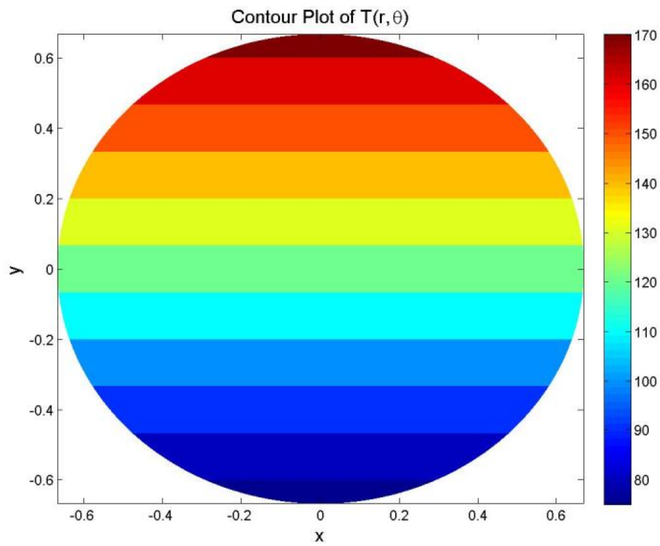

Fig. 2.9 Temperature solution for the pipe problem with $N = 65$.

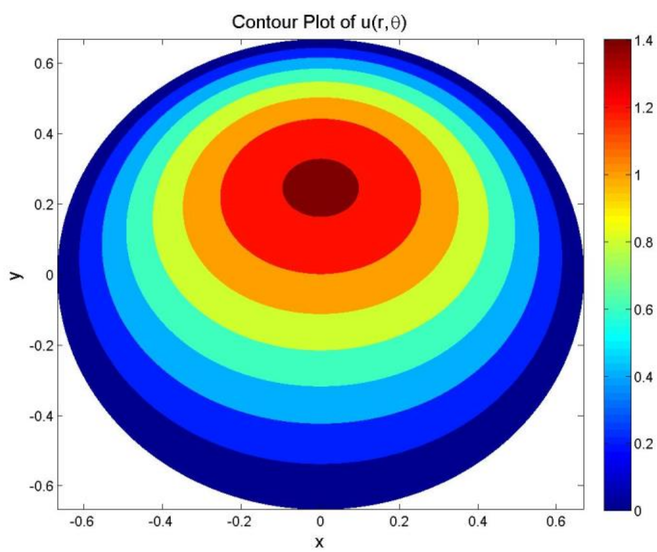

Fig. 2.10 Velocity solution for the pipe problem with $N = 65$.

Plots of the convergence history versus iterations or elapsed time should be provided for the temperature and velocity solutions. The calculation of $\Delta Q\%$ should be provided, and you should provide your thoughts on the (surprising?) answer. Finally, a copy of your code should be attached.