Course Project

535.421: Intermediate Fluid Dynamics

Overview

A proper introduction to the art of Computational Fluid Dynamics (CFD) would require several semesters on this topic alone. Clearly this cannot be accomplished in a first graduate course in fluid dynamics, where the semester is already filled with the necessary topics to understand the underlying principles of the subject. On the other hand, the two primary approaches for describing fluid flows that cannot be obtained from analytical solution of the governing equations are: the use of experiments and the use of the computer to find numerical solutions. We will speak of experimental techniques throughout the course in each module. The use of computers in fluid dynamics is so *pervasive* that to ignore this topic completely until a later course is not prudent. The compromise that has been adopted is to provide an introduction to CFD in Module 2 sufficient to allow the solution of two typical engineering problems contained in this course project. Students will develop grids, write computer codes, produce converged solutions, assess the error, plot flow fields and compute derived engineering quantities from the results. From this experience students will gain an appreciation for many of the primary features of CFD. This project will be graded on the basis of 100 points, and the grade will represent 25% of the total course grade.

A good introduction to CFD is gained by focusing attention on a subclass of fluid flow problems governed by elliptic partial differential equations that are steady in time and vary only in space. Examples of such equations are the Poisson and Laplace equations, which are the topic of the first part of the project. The point of the first project problem is to allow the student to develop the needed code to complete a CFD solution of a simple problem. After gaining experience with the first problem, portions of this code are reused, and the rest is expanded upon to tackle an engineering-oriented pipe flow problem which is the subject of the second part of the project. Students are required to complete both parts of the project. This document will concentrate on the assignment of the problems, submission details, required deliverables and a grading rubric. Substantial background information and advice on the approach to these problems may be found in the Module 2 Lecture Notes. Also, several of the discussion activities are designed to assist progress towards the solution.

Objectives

After completion of this project, you will be able to:

- Discretize the governing equation for a fluid flow problem by using central difference estimates for the derivatives
- Develop both rectangular and polar grids for the problems and assign boundary and symmetry conditions appropriately
- Compute solutions using two iterative methods: Jacobi and Gauss-Seidel
- Plot the convergence history as well as contour plots of the solution
- Produce a series of solutions, where preceding computations are required to define the next
- Compute derived quantities of engineering significance that permit appropriate business decisions to be made

Project Assignment – Part 1 – Solution of Elliptic Partial Differential Equations

Introduction

The diffusion of a scalar quantity φ in a two dimensional physical domain can be mathematically described by Poisson's equation

$$\nabla^2\varphi = f(x,y) \quad , \tag{1}$$

where ∇ is the gradient operator and $f(x,y)$ is a distribution of sources (or sinks) within the domain. A Poisson equation is an elliptic partial differential equation; by this it is meant that the behavior of the scalar quantity φ, at any given point in the physical domain, is related to all other values of φ within the domain, as well as to the boundary conditions and to the term $f(x,y)$. When the right-hand side of this equation is zero, then the equation is called a Laplace equation.

Part 1 Statement – Do the following two problems.

I. Consider the two-dimensional Laplace equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \quad , \tag{2}$$

on the square domain $0 \le x \le 1, 0 \le y \le 1$ with homogeneous boundary conditions

$$\varphi(0,y) = \varphi(1,y) = \varphi(x,0) = \varphi(x,1) = 0 \quad . \tag{3}$$

Discretize equation (2) on a square mesh with $\Delta x = \Delta y = 1/64$. Using random numbers distributed between 0 and 1 for the initial guess φ^0, invert the system of algebraic equations resulting from the discretization of equation (2) using both the Jacobi method and then the Gauss-Seidel method.

The L_2 norm of the error, defined as

$$\left\| \varepsilon^n \right\|_2 = \left[\sum_{i,j} (\varphi_{i,j}^n)^2 \right]^{\frac{1}{2}} \quad , \tag{4}$$

can be used as a measure of convergence. Choose a small number, $\varepsilon_{\min} = 1\times10^{-6}$, say, and cease iterating when

$$\Delta\varepsilon^n = \left| \; \left\| \varepsilon^n \right\|_2 - \left\| \varepsilon^{n-1} \right\|_2 \; \right| \le \varepsilon_{\min} \quad . \tag{5}$$

Plot $\Delta\varepsilon^n$ as a function of the number of iterations n and the elapsed time. These plots show the *convergence history* of the solution. Specifically, this will be two separate plots with two curves (one for each of the two iteration methods) on each plot. Alternatively, one curve on each plot for a total of four plots is acceptable. For each plot, the abscissa should be a linear axis, and the ordinate is best plotted on a logarithmic axis. Produce a contour plot showing the converged solution for φ. Comment on your results.

II. Repeat the basic problem for the two dimensional Poisson equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = f(x,y) \quad , \tag{6}$$

on the square domain $0 \le x \le 1, 0 \le y \le 1$ with boundary conditions

$$\varphi(0,y) = 1 \quad \varphi(1,y) = -1 \quad \varphi(x,0) = \varphi(x,1) = 0 \quad , \tag{7}$$

and with a source and a sink located at

Case a $\quad f(0.25,0.5) = 1 \quad f(0.75,0.5) = -1 \quad ,$ (8)

Case b $\quad f(0.25,0.5) = 2500 \quad f(0.75,0.5) = -2500 \quad ,$ (9)

Case c $\quad f(0.25,0.5) = 15000 \quad f(0.75,0.5) = -15000 \quad .$ (10)

For case (a) **only**, plot the two error plots as described above. For cases (a) - (c), create contour plots showing the converged solution for φ.

Project Assignment – Part 2 – Pipe Flow Problem

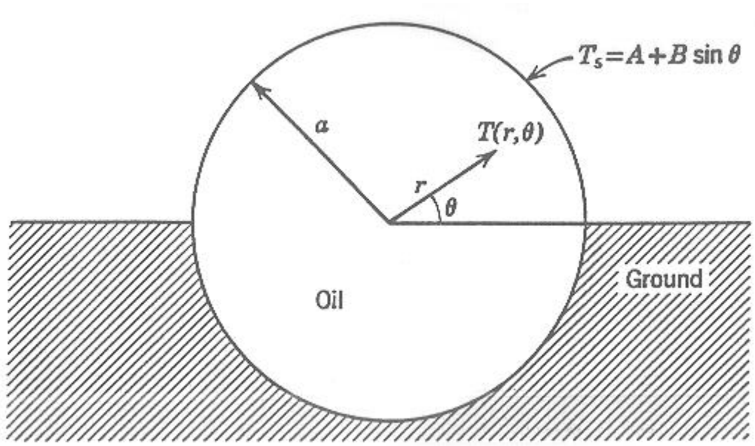

Fig. 1 Cross-sectional view of pipe flow problem

Figure 1 shows a cross section of an oil pipeline that is half buried in relatively cool ground. The top half of the pipe is exposed to the sun, and the surface temperature of the pipe may be approximated by

$$T_s = A + B\sin\theta \quad , \tag{11}$$

where A and B are constants given below. Assuming that the thermal conductivity of the oil is constant and that heat transfer across the section is by conduction only, the local oil temperature $T = T(r,\theta)$ is the solution of the simplified heat conduction equation in cylindrical coordinates:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \quad . \tag{12}$$

(a) Write a program that will perform a finite difference solution to find T as a function of r and θ. An iterative technique, such as the Gauss-Seidel method is suggested. Observe that symmetry makes it necessary to consider only the region $-\pi/2 \le \theta \le \pi/2$, at most. The input data should include values for A, B, k (the number of radial spacings Δr), l (the number of angular increments $\Delta\theta$ between $\theta = -\pi/2$ and $\theta = \pi/2$), M (the upper limit on the number of iterations to be performed) and ε (the tolerance used in testing for convergence). Note that a grid that is equally spaced in both directions, where N refers to the number of grid lines in each direction is perhaps easiest. Suggested values are: $A = 125\,°F$, $B = 50\,°F$, $N-1 = k = l = 64$, $M = 10{,}000$ and $\varepsilon = 10^{-5}$. You may wish to use smaller values initially when testing the code. Increasing $N-1$ to 128, 256, etc. will give a more precise answer at the expense of computation time. This is encouraged after a working code has been obtained, but is not required. The use of a data type with double precision is recommended.

(b) Assuming steady flow along the pipe, show that the axial velocity $u = u(r,\theta)$ of the oil obeys the following PDE:

$$\frac{dp}{dz} = \frac{1}{r}\frac{\partial}{\partial r}\left(\mu r \frac{\partial u}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\mu}{r}\frac{\partial u}{\partial \theta}\right) \quad , \tag{13}$$

where dp/dz is the pressure gradient along the pipe, and μ is the viscosity of the oil, which varies with temperature according to the empirical law

$$\mu = \mu_0\, e^{\frac{c}{T+d}} \quad , \tag{14}$$

where μ_0, c and d are constants given below.

Write an extension to the program developed in part (a) that will use values for dp/dz, μ_0, c, d and R given below, and proceed to compute the distribution of velocity u over the cross section, based on the temperatures already computed in part (a). The following values are suggested:

$$\frac{dp}{dz} = -0.05\frac{lb_f}{ft^3} \quad \mu_0 = 5\times10^{-9}\,\frac{lb_f\, s}{ft^2} \quad c = 8000\,°F \quad d = 460\,°F \quad R = \frac{2}{3}\,ft \quad .$$

(c) Finally, write an additional extension to the program that will estimate the actual flow rate of oil, Q_{act} ft^3/s by numerically evaluating the integral

$$Q_{act} = 2\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{R} u\, r\, dr\, d\theta \quad . \tag{15}$$

Compare Q_{act} with an estimated value Q_{est} that is obtained by assuming a Hagen-Poiseuille velocity profile at a constant viscosity evaluated at the mean temperature $A\,°F$:

$$Q_{est} = \frac{\pi R^4}{8\mu_0 e^{\frac{c}{A+d}}}\left(-\frac{dp}{dz}\right) \quad . \tag{16}$$

Express the answer in the form:

$$\Delta Q\ (\%) = \left| \frac{Q_{act} - Q_{est}}{Q_{act}} \right| \times 100 \quad . \tag{17}$$

Project Deliverables

Prepare a report describing Parts 1 and 2 of the project. Specific details regarding the format of the report are left to the discretion of the student; however, the following deliverables must be provided.

For the Laplace solution in Part 1, provide the convergence history plots and a contour plot of the solution. Comment on your results. For the Poisson solutions in Part 1, provide the convergence histories for case a, and contour plots of the solution for all three cases a through c. Comment on your results. Attach a copy of all code used to produce the solutions.

For Part 2 include the derivations of all discretized equations. These can be hand computations that have been scanned; they need not be typed. Provide convergence history plots for the temperature and velocity solutions. Prepare (at the minimum) contour plots based on the rectangular grid (see Module 2 for explanation) for the temperature, viscosity and velocity fields. However, the student is encouraged to produce *polar contour plots* of these solutions as they will be more meaningful, but this is not required. Comment on each of these solutions. Compute $\Delta Q\ (\%)$ and comment on the surprising result that is obtained. Attach a copy of all code used to produce the solutions.

Grouping

Students are not placed in groups for the completion of this project. Students may interact with other students as desired for advice and sharing of ideas; however, each student is expected to submit the project deliverables individually. Group activities that help facilitate the completion of this project will take place via several of the Discussion Activities.

Submission

The course project will be due no later than midnight of Day 6 in Module 14. Course Projects should be submitted in Blackboard as a Word or PDF document.

Plagarism

Plagiarism is defined as taking the words, ideas or thoughts of another and representing them as one's own. If you use the ideas of another, provide a complete citation in the source work; if you use the words of another, present the words in the correct quotation notation (indentation or enclosed in quotation marks, as appropriate) and include a complete citation to the source.

Grading Rubric

This rubric assumes the student shows all work for solving the project problems. This is to include hand computations, computer code printouts of numerical computations, plots documenting results, computations of derived quantities, and a commentary analyzing the results. Submissions not meeting this standard will have the grade adjusted downward, as appropriate. The use of the word *progress* in the table below implies work that displays an *understanding of the problem* and not just the quantity of writing that the student provides.

<table>
<tr><th colspan="4">Results</th></tr>
<tr><td></td><td>Poor</td><td>Good</td><td>Exceptional</td></tr>
<tr><td>Part 1
(35 points)</td><td rowspan="2">Minimal or no progress toward a solution.
(0-19 points)</td><td rowspan="2">Substantial progress toward a solution.
(20-29 points)</td><td rowspan="2">Student correctly solves the problem.
(30-35 points)</td></tr>
<tr><td>Part 2
(35 points)</td></tr>
<tr><th colspan="4">Presentation</th></tr>
<tr><td></td><td>Poor</td><td>Good</td><td>Exceptional</td></tr>
<tr><td>Graphs
(15 points)</td><td>Most graphs missing or incorrect
(0-5 points)</td><td>Some graphs missing or incomplete information
(6-10 points)</td><td>Followed directions and produced all required graphs
(11-15 points)</td></tr>
<tr><td>Analysis
(15 points)</td><td>Little or no explanation and/or little grasp of the results
(0-5 points)</td><td>Partially explains and/or shows a partial grasp of the results
(6-10 points)</td><td>Fully explains and demonstrates an understanding of the results
(11-15 points)</td></tr>
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Grade for Computer Project

The grade ranges listed below are typical and should be used for guidance. However, additional consideration will be given to those students that produce more accurate computations or more meaningful plots beyond the minimum requirements.

- 100-90 pts: A
- 89-80 pts: B
- 79-0 pts: C – F, (Determined on a case-by-case basis)

References

Anderson, D.A., Tannehill, J.C. and Pletcher, R.H. (1984). *Computational Fluid Mechanics and Heat Transfer*. New York, NY: Hemisphere Publishing Corp.

Burden, R.L. and Faires, J.D. (1981). *Numerical Analysis*. Boston, MA: Prindle, Weber and Schmidt.

Carnahan, B., Luther, H.A. and Wilkes, J.O. (1969). *Applied Numerical Methods*. New York, NY: John Wiley & Sons, Chapter 7, pp 429-530.

Hirsch, C. (1988). *Numerical Computation of Internal and External Flows*, Volume 1: Fundamentals of Numerical Discretization. New York, NY: John Wiley and Sons, ISBN 0-471-92385-0.