MATH 447/647: Probability Models
Fall 2020 Semester
Midterm 2: Due Wednesday November 25, 5:30pm ET.

Instructions

- Write your full name, “Midterm 2”, and the date at the top of the first page.
- Show all work, including each step of your solution, to earn maximal partial credit.
- Each question has multiple parts. Write legibly and neatly. Box your final answers.
- Use Genius Scan or a similar application to convert your solutions to .pdf format.
- Submit a single .pdf file to Gradescope under the assignment “Midterm 2”.
- If you have any questions, email me or come to office hours (WF 11:00am-12:00pm)

Assignment (2 Problems: 50 + 50 = 100 points total.)

☐ **Problem 1** Consider the Markov chain $\{X_n\}_{n=0}^{\infty}$ with infinite state space

$$\mathbb{X} = \{0, 1, 2, 3, 4, \ldots\}$$

and 1-step transition probabilities

$$\mathbf{P}_{ij} = \begin{cases} 0.9 & \text{if } j = i \\ 0.1 & \text{if } j = i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

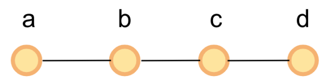
- 1.1 [10 points] Assuming $X_0 = 0$, calculate the time n marginal distribution of this Markov chain. Precisely, for each i in $\mathbb{X}$, calculate

  $$P(X_n = i \mid X_0 = 0).$$

  In this problem, both n and i are arbitrary.
- 1.2 [10 points] Assuming $X_0 = 0$, let T be the random time this Markov chain spends in the state 0 before jumping to a different state. Find the probability mass function of the discrete random variable T. Precisely, for each $n = 1, 2, 3, \ldots$, calculate

  $$P(T = n).$$

  In this problem, n is arbitrary.
- 1.3 [10 points] Find the communicating classes of the Markov chain.
- 1.4 [10 points] Determine if each communicating class is recurrent or transient.
- 1.5 [10 points] Is this Markov chain reversible or irreversible? Explain your reasoning.
  Hint: use Kolmogorov’s criterion



☐ **Problem 2** Consider the simple graph $\Gamma = (V, E)$ with 4 vertices and 3 edges above and let $\{X_n\}_{n=0}^{\infty}$ be the simple random walk on Γ as defined in HW5.

- 2.1 [10 points] Find the 1-step transition matrix $\mathbf{Q}$ for the simple random walk on Γ.
- 2.2 [10 points] Suppose $X_0 = a$. Let T be the first time at which the Markov chain returns to the state a. This *first return time* T is a discrete random variable

$$T = \min\{n \geq 1 \ : \ X_n = a\}.$$

  What is the probability that the chain visits the state d before time T?
- 2.3 [10 points] Find an equilibrium distribution for the Markov chain defined by $\mathbf{Q}$.
- 2.4 [10 points] Is the uniform distribution μ with

$$\vec{\mu} = [\mu_a \ \mu_b \ \mu_c \ \mu_d] = [\tfrac{1}{4} \ \tfrac{1}{4} \ \tfrac{1}{4} \ \tfrac{1}{4}]$$

  an equilibrium distribution of the Markov chain defined by $\mathbf{Q}$?
- 2.5 [10 points] Find the 1-step transition matrix $\mathbf{P}$ for the Hastings-Metropolis Markov chain which (i) has equilibrium distribution μ given by the uniform distribution above and (ii) has as its proposal chain the Markov chain defined by $\mathbf{Q}$ above.

☐ **Bonus** A *Markov chain in* $\mathbb{R}$ is a discrete time Markov chain $\{X_n\}_{n=0}^{\infty}$ in the continuum state space $\mathbb{X} = \mathbb{R}$. For most Markov chains in $\mathbb{R}$, the 1-step transition probabilities $\mathbf{P}_{xy}$ are all 0, so a new quantity is needed to describe these Markov chains. For any two states x, y in $\mathbb{R}$, the 1-*step transition density* is a function $\mathbf{K}(x, y)$ of (x, y) in $\mathbb{R}^2$ so that

$$P(X_{n+1} \text{ in } [a, b] \mid X_n = x) = \int_a^b \mathbf{K}(x, y)dy.$$

This $\mathbf{K}(x, y)$ is also called the *Markov kernel* or *stochastic kernel* of the Markov chain.

- X.1 [X points] Consider the chain $\{X_n\}_{n=0}^{\infty}$ in $\mathbb{X} = \mathbb{R}$ with 1-step transition densities

$$\mathbf{K}(x, y) = \begin{cases} 1 & \text{if } |y - x| \leq 0.5 \\ 0 & \text{otherwise} \end{cases}.$$

  Find $P(X_2 > 0.5 \mid X_0 = 0)$.
- X.2 [X points] Consider the chain $\{X_n\}_{n=0}^{\infty}$ in $\mathbb{X} = \mathbb{R}$ with 1-step transition densities

$$\mathbf{K}(x, y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2}}.$$

  Find $P(X_2 > 0.5 \mid X_0 = 0)$.