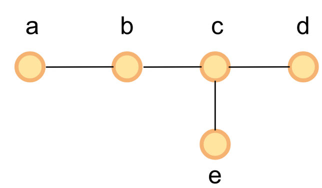
MATH 447/647: Probability Models
Fall 2020 Semester
Homework 6: Due Wednesday November 18, 5:30pm ET.

Instructions

- Write your full name, “Homework 6”, and the date at the top of the first page.
- Show all work, including each step of your solution, to earn maximal partial credit.
- Each question has multiple parts. Write legibly and neatly. Box your final answers.
- Use Genius Scan or a similar application to convert your solutions to .pdf format.
- Submit a single .pdf file to Gradescope under the assignment “Homework 6”.
- If you have any questions, email me or come to office hours (WF 11:00am-12:00pm)
- You are encouraged to work together (on Piazza) but must write up your own solutions.

Assignment (4 Problems: 20 + 30 + 20 + 30 = 100 points total.)

☐ **Problem 1** Consider the simple graph $\Gamma = (V, E)$ with 5 vertices and 4 edges below



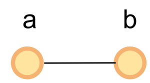
and let $\{X_n\}_{n=0}^{\infty}$ be the simple random walk on Γ as defined in HW5.

- 1.1 [10 points] Find an equilibrium distribution of this Markov chain.
- 1.2 [10 points] Is this Markov chain reversible?

☐ **Problem 2** In L14, we considered the experiment of selecting 1 of 4 indistinguishable molecules in two urns A and B uniformly at random, taking the chosen molecule out of its urn, and placing it in the opposite urn. We saw that the resulting stochastic process is a Markov chain in $\mathbb{X} = \{0, 1, 2, 3, 4\}$ with 1-step transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

- 2.1 [15 points] Is this Markov chain reversible?
- 2.2 [15 points] If initially all 4 molecules are in urn A, what is the probability that there are infinitely-many instances in which all 4 molecules are back in urn A?



☐ **Problem 3** Consider the simple graph $\Gamma = (V, E)$ with 2 vertices and 1 edge above and consider the probability distribution μ on $V = \{a, b\}$ given by

$$\vec{\mu} = [\mu_a \quad \mu_b] = [0.3 \quad 0.7].$$

- 3.1 [10 points] For this μ, find the 1-step transition matrix $\mathbf{P}$ of the Hastings-Metropolis Markov chain $\{X_n\}_{n=0}^{\infty}$ on $\mathbb{X} = V = \{a, b\}$ assuming that the proposal chain $\mathbf{Q}$ is the simple random walk on Γ. *Hint: to check your work, see if μ above satisfies the full balance equations $\vec{\mu} = \vec{\mu}\,\mathbf{P}$ or detailed balance equations $\mu_i \mathbf{P}_{ij} = \mu_j \mathbf{P}_{ji}$ for your $\mathbf{P}$.*

- 3.2 [10 points] Assume $X_0 = a$. What is the time $n = 5$ marginal distribution of the Hastings-Metropolis chain? Specifically, what are $P(X_5 = a)$ and $P(X_5 = b)$?

☐ **Problem 4** Consider the Markov chain on $\mathbb{X} = \{a, b, c\}$ with 1-step transition matrix

$$\mathbf{Q} = \begin{bmatrix} 0 & 0.9 & 0.1 \\ 0.1 & 0 & 0.9 \\ 0.9 & 0.1 & 0 \end{bmatrix}$$

- 4.1 [10 points] Is this Markov chain reversible?

- 4.2 [10 points] Consider the uniform distribution μ on $\{a, b, c\}$ given by

  $$\vec{\mu} = [\mu_a \quad \mu_b \quad \mu_c] = [\tfrac{1}{3} \quad \tfrac{1}{3} \quad \tfrac{1}{3}].$$

  Construct the 1-step transition matrix $\mathbf{P}$ for the Hastings-Metropolis Markov chain $\{X_n\}_{n=0}^{\infty}$ on $\mathbb{X} = \{a, b, c\}$ assuming that the proposal chain is $\mathbf{Q}$ above.

- 4.3 [10 points] Is your Hastings-Metropolis chain in Problem 4.2 reversible?

☐ **Bonus** [X points] Is the simple random walk on the graph below reversible?