642:550 — Homework 7

Unless otherwise noted, show all your steps.

Problem 1. Consider

$$v_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad v_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Apply the Gram–Schmidt algorithm to these vectors to obtain an orthonormal basis f_1, f_2, f_3.

Problem 2 and 3. Consider the *two dimensional* vector space $V = \mathrm{span}(v_1, v_2)$, where v_1, v_2 are given in Problem 1 above. Consider the projection of vectors in $\mathbb{R}^4$ onto this space V. Use two different methods to find the matrix P of this projection.

Problem 2.

Method I) Follow the procedure described in Question 2 (3) of the in-class questions on Oct 20 to find the projection matrix P.
Link: https://rutgers.instructure.com/courses/74179/files/12323143/download?wrap=1

Problem 3.

Method II) Utilize an orthonormal basis to find the projection matrix P.
In Problem 1 above, you've found an orhthonormal basis f_1, f_2 for the space $V = \mathrm{span}(v_1, v_2) = \mathrm{span}(f_1, f_2)$. Given a generic $y \in \mathbb{R}^4$, let $y' = Py$ denote the projected vector. Express y' as a linear combination as

$$Py = y' = b_1 f_1 + b_2 f_2 \tag{*}$$

(1) Use the fact that f_1, f_2 are orthonormal to find b_1 b_2 in terms of f_1, f_2, and y.
Instruction: operator with symbols f_1, f_2, y. Do not insert the values of f_1, f_2 yet.

(2) Insert your answer in (1) back into (*). Identity the matrix P from your result. Express your answer for P in terms of f_1, f_2.
Instruction: operator with symbols f_1, f_2. Do not insert the values of f_1, f_2 yet.

(3) Now insert in the numerical values of f_1, f_2 (from Problem 1 above) into your answer for P in (2), and check that the result agrees with what you found in Problem 2.

Problem 4. Given a vector space V in $\mathbb{R}^n$. Consider the projection of vectors in $\mathbb{R}^n$ onto V, and let P denote the projection matrix. Follow the steps below to prove that $P^2 = P$ and $P^{\mathrm{T}} = P$.

(1) Let $f_1, \ldots, f_k$ be an orthonormal basis for V. (Such an orthonormal basis always exists. We won't worry about proving its existence here.) Express the projection matrix P in term of $f_1, \ldots, f_k$.
Hint: generalize the procedure in Problem 3 above.
Instruction: explain all your derivation. Simply showing the answer does not account for a full solution.

(2) Use your answer in (1) to verify $P^2 = P$ and $P^{\mathrm{T}} = P$.

Remark: In Problem 4 of Homework 6, you've shown that if $P^2 = P$ and $P^{\mathrm{T}} = P$ then P is a projection matrix. Problem 3 here gives the converse statement. Combining these two statements gives

"A square matrix P is a projection matrix if and only if $P^2 = P$ and $P^{\mathrm{T}} = P$."