(1) Consider the parametric vector equation

$$\vec{x} = \langle 1+t, 2+t, 3-t \rangle.$$

What position will $\vec{x}$ be at when $t = 0$? What about $t = 1$?

(2) Write the parametric vector equation of a line that passes through the point $(1, 3, 4)$ at $t = 0$, having velocity vector $\langle -1, 2, -1 \rangle$.

(3) To find the distance between a point, Q, and a line, ℓ, we need to construct the vector $\vec{PQ}$ where P is *any* point on line ℓ.

Suppose ℓ is given parametrically by

$$\vec{x} = \langle 1-2t, 5+3t, 4-t \rangle.$$

(a) What is an obvious choice for point P?

(b) If $Q = (1, 6, 5)$, determine $\vec{PQ}$ and find its cross product with the velocity vector, $\vec{v}$, for the line ℓ.

(c) Since the distance between Q and ℓ is given by $|\vec{PQ}| \cdot |\sin\theta|$ (where θ is the angle between $\vec{PQ}$ and $\vec{v}$), we can find the distance using the cross product in the previous problem. Do it.

(4) Suppose a line is given parametrically by

$$\vec{x} = \langle 2t - 1, 7t, 6t + 5 \rangle.$$

Find the distance between this line and the point $(2, 5, 13)$.

(5) When we want to find the distance between a point and a plane, we use a dot product with the normal vector for the plane. Consider the plane whose scalar equation is

$$2x + 3y - z = 10.$$

The ingredients we'll need for computing the distance from some point Q to this plane are:

(a) A normal vector for the plane. ()The coordinates of this vector are literally visible in the equation.) What is it?

(b) A second point that lies in the plane. Often this can be determined by setting two of the variables to 0. Since we just need *any* point in the plane, it's a good idea to choose which variables to set to 0, in such a way that the calculation is easy. Find a nice point that lies in this plane.

(6) Find the distance from the plane given in the previous problem to the point $Q = (3, 4, 5)$.

(7) Two planes will generally intersect in a line. If their normal vectors point in the same direction they may be parallel. Parallel planes can even be literally the same, even though their equations appear to be different! But, if the normal vectors are not multiples of one another, the planes will intersect in a line.

Identify the following situations (there is one of each) Parallel planes, Identical planes, Planes that intersect in a line.

(a) $3x + 4y - 5z = 9$ and $-6x - 8y + 10z = -18$

(b) $3x - 2y + z = 2$ and $4x + y - z = 4$

(c) $2x + 4y + 6z = 9$ and $2x + 4y + 6z = 12$

(8) When two planes are not parallel, a vector that lies in their intersection will be perpendicular to both of their normal vectors. This allows us to use a cross product to find a vector that can be used as the velocity vector in finding a parametric description for the line of intersection.

Find a parametric equation for the line of intersection of

$$x - y + 2z = 5 \qquad \text{and} \qquad 2x + 3y - z = 5.$$

Hint: $(1, 2, 3)$ is a point that lies in both planes.

(1) Generalized cylinders have equations that don't involve one of the variables. They will run parallel to the missing variable's axis and have a cross-section determined by the equation.

For example $y - x^2 = 0$ is a generalized cylinder with a parabolic cross-section that runs parallel to the z-axis.

Identify each of the following cylinders.

(a) $x^2 + z^2 = 4$

(b) $y^2 - z^2 = 1$

(c) $x - z^2 = 0$

(d) $y^2 + x^2 = 9$

(e) $y^3 - x = 0$

(2) A way to get started on figuring out the graph of an arbitrary quadratic polynomial in 3 variables is to determine where (or if) it intersects the axes. To do this you set the other two variables to 0 and solve. These points are the *intercepts* – there may be 2, 1, or 0 intercepts on each axis.

What are the intercepts of the following?

(a) $x^2 + y^2 - z = 4$

(b) $x^2 - y^2 - z = 9$

(c) $x^2 - y^2 + z^2 = 16$

(d) $y^2 + x^2 = 9$

(e) $(x/3)^2 + (y/4)^2 + (z/2)^2 = 1$

(3) Occasionally, the process of finding the intercepts on an axis fails spectacularly – when the graph contains the entire axis. Which axis is wholly contained in the graph of $y - z^2 = 0$?

(4) The next step in identifying a quadric surface is to determine the traces – the plane curves where the surface intersects the coordinate planes. To do this you need to be able to recognize the forms of plane curves in two variables. What are the following? (Draw sketches.)

(a) $x^2 + y^2 = 4$

(b) $x^2 + y^2 = -4$

(c) $x^2 - y = 0$

(d) $y^2 - x^2 = 1$

(e) $x^2 - y^2 = 1$

(f) $x^2 - y^2 = 0$

(5) What are the traces of the following? Specify a curve (which may be empty or non-existent!) for each of the xy, xz and yz planes. (You'll need to set one variable to zero – for instance to find the trace in the xz plane, set $y = 0$.)

(a) $x^2 + y^2 + z^2 = 4$

(b) $x^2 - y^2 - z = 9$

(c) $x^2 + y^2 - z^2 = 1$

(d) $y^2 + x^2 = 9$

(e) $x^2 - y^2 - z^2 = 1$

(6) Suppose you determine that a graph has traces in the xz and yz planes that are parabolas and that it's trace in the xy plane is a circle. Which kind of quadric surface is it?

(7) A quick way to distinguish the two kinds of hyperboloids (1-sheet and 2-sheet) is that the hyperboloid of two sheets has one trace that is non-existent. For instance $x^2 + y^2 - z^2 = -1$ misses the xy plane entirely. In looking at the equation (with $z - 0$ we see a sum of squares that needs to be negative which is impossible.

Each of the following is a hyperboloid of two sheets. Which coordinate planes have these "empty" traces?

(a) $x^2 + y^2 - z^2 = -1$

(b) $x^2 - y^2 + z^2 = -1$

(c) $x^2 - y^2 - z^2 = 1$

(8) Describe the object in three-dimensional space defined by the equation $3x + 2y - z = 11$.

(9) Describe the object in three-dimensional space defined by the equation $x^2 + y^2 + z^2 = 25$.

(10) Describe the object in three-dimensional space defined by the equation $x^2 - y = 0$.

(11) Describe the object in three-dimensional space defined by the equation $xy - z = 0$.

(12) Describe the object in three-dimensional space defined by the equation $x^2 - y^2 + z^2 = 4$.

(1) Write a vector whose components are functions of a parameter t that will define a spacecurve through the point $(1, 2, 5)$ which has velocity vector $\langle 1, 1, -1 \rangle$.

(2) Create a space curve (a vector whose components are functions of a parameter t) that defines a helix with radius 2 whose central line is the z-axis.

(3) If the curve you just defined were interpreted as the velocity vectors of another curve, what would that other curve be?

(4) Find the derivative of $\langle 3, \cos t, \sin t \rangle$.

(5) Find the derivative of $t^3 \cdot \langle \cos t, \sin t, \sin 2t \rangle$

(6) Find the derivative of $\langle t, t^2, t^3 \rangle \cdot \langle 1, 2t, 3t^2 \rangle$

(7) Find the derivative of $\langle t, t^2, t^3 \rangle \times \langle t, t, t \rangle$

(1) The function

$$f(x,y) \;=\; \frac{(x+y)^2}{x^2+y^2}$$

is undefined at $(0,0)$. Compute the limit as (x,y) approaches $(0,0)$ in two ways:

along the line $y = x$ and along the line $y = -x$.

(2) Here are a few functions to practice taking partial derivatives with. (For each function $f(x,y)$ compute both f_x and f_y.)

(a) xy^3

(b) e^{xy}

(c) $sin(x+y)$

(d) x^2ye^y

(3) Compute the mixed second order partial derivatives f_{xy} and f_{yx} for the following functions. By Clairout's theorem the answers will be identical, but show the process and verify that $(f_x)_y$ is equal to $(f_y)_x$ in each case.

(a) x^3y^3

(b) e^{xy}

(c) $cos(3x + xy + y)$

(d) $x^2 - 2xy + y^2$

(4) The partial derivatives (evaluated at a point) give the slope of a tangent line to the surface $z = f(x, y)$ at that point – either traveling in the direction of the x-axis (for f_x) or in the direction of the y-axis (for f_y). Use this to create parametric representations for the two lines through $(1, 2, 2)$ that are tangent to the surface $z = xy$.

(5) Consider the surface given by $z = f(x, y) = x^2 - y^2$.

(a) If we restrict our position to the line $y = 1$ in the xy-plane we get something that may be viewed as solely being a function of x. Sketch a graph of $z = f(x, 1)$. What is the derivative of this function when $x = 1$?

(b) If instead, we restrict our position to lie on the line $x = 1$ in the xy-plane we get something that may be viewed as solely being a function of y. Sketch a graph of $z = f(1, y)$. What is the derivative of this function when $y = 1$?

(6) Calculate the partial derivatives of $f(x, y) = x^2 - y^2$ with respect to x and y.

$f_x(x, y) =$

$f_y(x, y) =$

Evaluate the above at $x = y = 1$ and compare to the derivatives in problem 5.

(1) Functions that have one input and two outputs, and functions that have two inputs and one output, can be composed in two different ways. Make black box diagrams for both compositions.

(2) Which composition above yields a function that has an ordinary derivative, and which gives a function with partial derivatives?

(3) Draw the dependency diagram with the inputs, hidden variables and output for the composition

$$z = f(\vec{r}(t)), \quad \text{where } f(x, y) = x^2 + y, \text{ and } \vec{r}(t) = \langle t, t^2 \rangle.$$

(4) Use the multi-variable chain rule to find the derivatives (or partial derivatives, when appropriate) for the following.

(a) $f(\vec{r}(t))$, where $f(x, y) = x^2 + y^2$ and $\vec{r}(t) = \langle 3t, 4t \rangle$.

(b) $\vec{r}(f(x, y))$, where $f(x, y) = e^{xy}$ and $\vec{r}(t) = \langle \cos t, \sin t \rangle$.

(c) $f(g(s, t), h(s, t)))$, where $f(x, y) = \cos x \cdot \sin y$ and $g(s, t) = 2t - s$ and $h(s, t) = t + 2s$.

(5) Each member of your group must do the following: Create a function of two variables that requires the product rule to find the partial derivative w.r.t. x, but does not require the product rule to find the partial derivative w.r.t. y.

Then, everyone swaps with someone and "does" the derivatives.

(6) Illustrate how the product rule can be deduced from the multi-variable chain rule. (Use the composition of the function $p(x, y) = xy$ with two arbitrary functions of a single variable, $f(t)$ and $g(t)$.)

(7) Deduce the quotient rule from the multi-variable chain rule using the function $q(x, y) = x/y$.

(1) Determine an equation for the tangent plane to $z = x^2 + y^2$ at the point $(1, 1, 2)$.

(2) Determine an equation for the tangent plane to $z = x^2 - y^2$ at the point $(2, 1, 3)$.

(3) The differential dz is defined in the book as

$$\mathrm{d}z \;=\; f_x(x,y)\mathrm{d}x \;+\; f_y(x,y)\mathrm{d}y.$$

Re-express this as a dot product.

(4) Calculate dz for the surface $z = \cos(x)\sin(y)$.

(1) The *Lagrange conditions* are the set of scalar equations that we get from the vector equation $\vec{\nabla} f = \lambda \vec{\nabla} g$. If $f(x, y) = x^2 + 2y^2$ and the constraint curve is given implictly by $x^2 + 2xy + y^2$, what are the Lagrange conditions?

(2) Sometimes the constraint is given explicitly. In such cases we must re-express the constraint in an implicit form. For example,

> What point on the parabola $y = x^2$ produces the minimum value for the function $f(x, y) = 2x + y$?

What is $g(x, y)$?
Solve the problem.

(3) A very common problem type that uses the Lagrange multipliers method is: What point along the curve $g(x, y) = k$ is closest to the point (a, b). In this scenario the distance between (x, y) and (a, b) is the objective, but it is often easier to use the square of this distance.

For example, to find the point on $y = 1/x$ that is closest to $(1, 2)$ we would convert the curve to $xy = 1$ and use $(x-1)^2+(y-2)^2$ as the objective function.

Set up (but don't solve) the system.

(4) Find the absolute maximum and minimum values of $f(x, y) = x + y$ on the ellipse $x^2 + xy + y^2 = 1$.

(5) Find the extreme values of $f(x, y) = x^2 + y^2 + 2$ on the ellipse $x^2 + xy + y^2 = 4$.

(6) Here's one in 3-d.

What points on the cone $z^2 = x^2 + y^2$ are closest to the point $(1, 1, 0)$?

(1) Sometimes a constrained maximum/minimum problem can be solved by Lagrange multipliers and by a direct approach. Suppose we want to find the maximum and minimum values of $f(x, y) = xy$ with the constraint that (x, y) lies on the unit circle $x^2 + y^2 = 1$.

(a) First let's solve this using the Lagrange multipliers technique. We have $f(x, y) = xy$ and $g(x, y) = x^2 + y^2$.

(b) The other approach is to parameterize the constraint curve, for the unit circle this is easy:

$$\vec{r}(t) = \langle \cos t, \sin t \rangle$$

We then look at the composition $f(\vec{r}(t))$ and find it's min/max values by the methods we learned in Calc I. (This composition is, in the final analysis, a function with a single input and output.) A final step (since our answer will be in terms of t) is to convert the t values to the corresponding (x, y) pairs.

(2) Often we need to combine methods - use the ideas of section 15.7 to locate extreme values on the interior of a region and Lagrange multipliers to find extreme values on the boundary of a region. Suppose we wish to find the maximum and minimum values of

$$f(x, y) = x^2 + y^2 - 2x - 4y + 6$$

on the closed disk $x^2 + y^2 \leq 9$.

(a) Use Lagrange multipliers to find the extrema on the boundary of the disk.

(b) Find critical points by solving $\vec{\nabla} f = \vec{0}$.

(c) Evaluate $f(x, y)$ at the points discovered in parts (a) and (b).

(d) What are the absolute maximum and minimum of $f(x, y)$ on this disk?

(3) If the region we are concerned with has piece-wise boundary curves we will also have to look at the boundaries of the boundaries!

Consider the function $f(x, y) = x^2+y^2$. Suppose we want to find the extreme values of $f(x, y)$ on the square region bounded by $x = 1$, $x = -1$, $y = 1$ and $y = -1$.

(a) We can parameterize the right-hand boundary of the square as $\langle 1, t \rangle$ where $-1 \leq t \leq t$. Use Calc I techniques to find the min/max values of the composition of f with this parameterization. Remember to consider the endpoints of the interval $t = \pm 1$.

(b) Use Calc III methods (setting the gradient equal to zero and solving for critical points) to find extrema on the interior of the square.

(c) Finally, we should make an argument that the other extrema can be determined by the symmetries of this problem. What are the four locations where the absolute maximums are found? Where is the unique absolute minimum?

(4) Find the absolute extreme values ()and the locations where they occur), for $f(x, y) = \cos x \cdot \cos y$ over the square region satisfying $-3\pi/4 \leq x \leq 3\pi/4$ and $-3\pi/4 \leq y \leq 3\pi/4$.

(1) When we integrate a function of two variables with respect to one of them we treat the other variable (*and anything that is strictly a function of the other variable*) as a constant. Find the following "partial integrals."

(a) $\int_0^1 xy \, \mathrm{d}x$

(b) $\int_1^2 xy^2 \, \mathrm{d}y$

(c) $\int_0^\pi x \cos xy \, \mathrm{d}x$

(2) Let R be the region in the x-y plane where $1 \leq x \leq 4$ and $2 \leq y \leq 3$. We can find the volume of the 3-dimensional region that lies above R and below the surface $z = f(x, y)$ in two different ways using iterated integrals. Setup both integrals.

(3) Suppose $f(x, y) = x^2 y$ and R is the region given in problem 2. Verify that Fubini's theorem holds in this instance. In other words compute both of the iterated integrals above and check that they give the same result.

(4) The surface $z = xy$ (a hyperbolic paraboloid) lies above the $x - y$ plane in quadrants I and III, and below it in quadrants II and IV. If R is a rectangular region centered at the origin, does it stand to reason that

$$\iint xy \, \mathrm{d}A \;=\; 0?$$

Verify this where R is the square with side length 2 centered at the origin.

(5) Consider the volume in 3-dimensional space that lies above the region R, where

$$R = \{(x, y) \mid 2 \leq x \leq 3 \text{ and } 2 \leq y \leq 3\},$$

that is bounded above by the surface $z = x + y$.

What is its volume?

(6) Evaluate the following double integral by whichever iterated integral seems easier.

$$\iint_R ye^{xy}\ \mathrm{d}A$$

(7) Suppose x and y both lie between 5 and 6. Suppose further that $z = xy$. Clearly z lies between 25 and 36. Use a double integral to show that the average value of z is not $(25 + 36)/2 = 30.5$.

(1) If we have a double integral where the region of integration is not a rectangle we must obey the following:

- The outer integral must have constants in the limits.
- The inner integral can have functions of the other variable in the limits

Suppose the region R is the area bounded by the curves $y = 2 - x^2$ and $y = x^2$. Set up the integral that would find the volume of the solid between the $x - y$ plane and a surface $z = f(x, y)$ over that region.

(2) Consider the region bounded by the lines $y = 1$, $y = x$ and $y = 8 - x$. Depending on the order of integration we may have to split the region into two separate integrals or do it in a single integral. Determine the limits and the order of integration for both.

(3) We can use double integrals to find the area of a region. This is an example of the "integrate differential whatever to get whatever" approach.

$$\text{Area of } R = \iint_R \mathrm{d}A$$

or

$$A = \int_a^b \int_{g(x)}^{f(x)} \mathrm{d}y\ \mathrm{d}x.$$

(You may want to think of the integrand as being 1.)
Use a double integral to find the area of the region in problem 1.