Università della Svizzera italiana | Faculty of Economics

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FINANCIAL ECONOMETRICS – Homework 2

Due date: Tuesday October 27, 16:00 hrs

PROBLEM SETS FOR CLASSICAL REGRESSION

1. **Simple Linear Model**

   Consider the following simple linear model:

   $$y_i = \alpha + x_i'\beta + \varepsilon_i, \qquad \text{for } i = 1, ..., n \tag{1}$$

   where $y_i, \alpha, \varepsilon_i$ are scalars, $x_i = \begin{bmatrix} x_{i,1} \\ x_{i,2} \end{bmatrix}, \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$.

   Recall one of the assumptions on the OLS data generating process is:

   Assumption 3: the error terms are exogenous to the independent variables:

   $$E[\varepsilon | x_1, ..., x_n] = 0$$

   (a) Write (1) in a matrix form:

   $$y = X\gamma + \varepsilon,$$

   where $\gamma = \begin{bmatrix} \alpha \\ \beta_1 \\ \beta_2 \end{bmatrix}$ and give y, X, ε and their dimensions.

(b) Let's define

$$x^1 = \begin{bmatrix} x_{1,1} \\ \vdots \\ x_{n,1} \end{bmatrix}, \quad x^2 = \begin{bmatrix} x_{1,2} \\ \vdots \\ x_{n,2} \end{bmatrix}$$

Show that

$$E[\varepsilon] = 0, \quad E[{x^1}'\varepsilon] = 0 \quad E[{x^2}'\varepsilon] = 0$$

Hint: Law of Iterated Expectation

(c) **Unbiasedness:** In statistics, an estimator $\hat{\gamma}$ of parameter γ is unbiased if $E[\hat{\gamma}] = \gamma$. Prove the OLS estimator is unbiased.

Hint: Use Assumption 3 and Law of Iterated Expectation.

(d) Suppose there is a random variable z. Its realizations are observed as $\{z_i\}, i = 1, ..., n$. The sample average of z is defined as $\bar{z} := \frac{1}{n}\sum_{i=1}^{n} z_i$. As an approximation, let's assume here that $\bar{z} = E[z]$ when $n > N$. Use the result from (b) to show that

$$X'\varepsilon = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ for } n > N$$

2. **Vector Differentiation**

Let $f(b)$ be a function of a random vector b.

For questions (a) to (c), we have:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

(a) Let $f(b) = Ab$. Calculate $\dfrac{\partial f(b)}{\partial b}$.

(b) Let $f(b) = b'Ab$. Calculate $\dfrac{\partial f(b)}{\partial b}$.

(c) Let C be a 2×2 constant matrix, α be a constant and $f(b) = (b'Cb)^{\alpha}$.
Calculate $\dfrac{\partial f(b)}{\partial b}$.

Hint: Read A.8 of Appendix A of the Greene's book and use the Chain Rule.

3. **Multivariate Normal Distribution**

An $n \times 1$ random vector ε follows a multivariate normal distribution:

$$\varepsilon \sim N(0, \sigma^2 I_n),$$

where I_n represents an $n \times n$ identity matrix.

Suppose there is another $n \times 1$ random vector ξ, which is a linear transformation of vector ε:

$$\xi = C + D\varepsilon,$$

where C is an $n \times 1$ constant vector and D is an $n \times n$ constant orthogonal matrix with $D'D = I_n$.

(a) Give $E\left[\varepsilon_i\right], E\left[\varepsilon_i^2\right]$ and $E\left[\varepsilon_i \varepsilon_j\right]$ for $i, j = 1, ..., n$ and $i \neq j$.

(b) Write down the distribution of vector ξ.

Hint: The variance-covariance matrix of a $m \times 1$ vector x is given by: $V[x] = E[(x - E[x])(x - E[x])'] = E[xx'] - E[x]E[x]'$

(c) Calculate $E[\xi'\xi]$.

Hint: If z is a univariate random variable and $z \sim N(0, 1)$, then $E(z^2) = V(z) = 1$.