MATH 447/647: Probability Models
Fall 2020 Semester
Midterm 1: Due Thursday October 15, 5:30pm ET.

Instructions

- Write your full name, "Midterm 1", and the date at the top of the first page.
- Show all work, including each step of your solution, to earn maximal partial credit.
- Each question has multiple parts. Write legibly and neatly. Box your final answers.
- Use Genius Scan or a similar application to convert your solutions to .pdf format.
- Submit a single .pdf file to Gradescope under the assignment "Midterm 1".
- If you have any questions, email me or come to office hours (WF 11:00am-12:00pm)

Assignment (5 Problems: 20+ 20 + 20 + 20 +20 = 100 points total.)

☐ **Problem 1** [20 points] An airline has determined that over the past year, 5% of the people who purchase a ticket for a weekly flight from Logan to JFK will not show up. Their policy is to sell 52 tickets for this flight even though the plane can only hold 50 passengers. Assuming that whether or not a ticketed passenger shows up are independent events, what is the probability that there will be a seat available for every passenger who shows up?

☐ **Problem 2** An experimentalist flips six fair coins $C_1, C_2, C_3, C_4, C_5, C_6$ independently, each with outcome H or T. Let X_i be the random variable defined by

$$X_i = \begin{cases} 1 & \text{if} \quad C_i = H \\ -1 & \text{if} \quad C_i = T \end{cases}$$

and define two further random variables in terms of X_i by $Y = X_1 + X_2$ and $Z = X_1 - X_2$.

- 2.1 [10 points] Determine the probability that

$$\frac{X_1 + X_2 + X_3 + X_4 + X_5 + X_6}{6} = 0.$$

- 2.2 [5 points] Are Y and Z independent?
- 2.3 [5 points] Calculate $\text{Cov}[Y, Z]$.

☐ **Problem 3** Let X and Y be continuous random variables with joint pdf

$$f_{X,Y}(x, y) = \frac{1}{4\pi}\, e^{-\frac{x^2}{8} - \frac{y^2}{2}}$$

- 3.1 [10 points] Are X and Y independent?
- 3.2 [10 points] Compute $\text{Var}[X + Y]$.

☐ **Problem 4** Consider the planar domain Ω in $\mathbb{R}^2$ given by

$$\Omega = \{\vec{v} = (x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 1\}$$

the closed disk of radius 1 centered at the origin in the plane. Let $\vec{V} = (X, Y)$ be a vector in Ω chosen uniformly at random according to P_{unif} from Lecture 6 with joint pdf

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi} & \text{if } \ (x, y) \in \Omega \\ 0 & \text{if } \ (x, y) \notin \Omega \end{cases}$$

- 4.1 [10 points] What is $f_X(x)$?
- 4.2 [10 points] Are X and Y independent?

☐ **Problem 5** Let X_2 and X_4 be the Bernoulli random variables from Lecture 7 which describe the position of the ball in the Galton board after 2 or 4 steps, respectively.

- 5.1 [10 points] What is $P(X_4 = 2|X_2 = 2)$?
- 5.2 [10 points] What is $P(X_2 = 2|X_4 = 2)$?

☐ **Bonus** [Discovering Benford's Law]

- X.1 [X points] If U is a uniform random variable on the interval $[7, 8]$, raising U to the 10th power defines another random variable $Y = 10^U$ on the interval $[10^7, 10^8]$. This new random variable Y is not uniform. For each $d = 1, 2, 3, 4, 5, 6, 7, 8, 9$, calculate

  $$p(d) = P(\text{ the first digit of } Y \text{ is } d)$$

  For example, the first digit of 5551234 is 5.

- X.2 [X points] Watch S1E4 "Digits" of *Connected* on Netflix – or read any articles online or in print – on Benford's Law and discuss any example that interests you.