MATH 447/647: Probability Models
Fall 2020 Semester
Homework 4: Due Thursday October 22, 5:30pm ET.

Instructions

- Write your full name, "Homework 4", and the date at the top of the first page.
- Show all work, including each step of your solution, to earn maximal partial credit.
- Each question has multiple parts. Write legibly and neatly. Box your final answers.
- Use Genius Scan or a similar application to convert your solutions to .pdf format.
- Submit a single .pdf file to Gradescope under the assignment "Homework 4".
- If you have any questions, email me or come to office hours (WF 11:00am-12:00pm)
- You are encouraged to work together (on Piazza) but must write up your own solutions.

Assignment (5 Problems: 20 + 20 + 20 + 20 + 20 = 100 points total.)

☐ **Problem 1** The Markov chain $X = \{X_n\}_{n=0}^{\infty}$ has state space $\mathbf{X} = \{a, b\}$ and

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.1 \\ 0.02 & 0.98 \end{bmatrix}$$

- 1.1 [5 points] Compute $P(X_2 = a|X_0 = a)$.
- 1.2 [5 points] Compute $P(X_3 = b|X_0 = a)$.
- 1.3 [5 points] Is this Markov chain irreducible?
- 1.4 [5 points] Is the state b recurrent?

☐ **Problem 2** The Markov chain $X = \{X_n\}_{n=0}^{\infty}$ has state space $\mathbf{X} = \{a, b, c\}$ and

$$\mathbf{P} = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

- 2.1 [5 points] Compute $P(X_2 = a|X_0 = a)$.
- 2.2 [5 points] Compute $P(X_3 = c|X_0 = b)$.
- 2.3 [5 points] Is this Markov chain irreducible?
- 2.4 [5 points] Is the state b recurrent?

☐ **Problem 3** The Markov chain $X = \{X_n\}_{n=0}^{\infty}$ has state space $\mathbf{X} = \{a, b, c\}$ and

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.6 & 0.2 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$$

- 3.1 [5 points] Compute $P(X_2 = a | X_0 = a)$.
- 3.2 [5 points] Compute $P(X_3 = c | X_0 = b)$.
- 3.3 [5 points] Is this Markov chain irreducible?
- 3.4 [5 points] Is the state b recurrent?

☐ **Problem 4** The Markov chain $X = \{X_n\}_{n=0}^{\infty}$ has state space $\mathbf{X} = \{a, b, c, d, e\}$ and

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 & 0 \\ 0.25 & 0.5 & 0.25 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

- 4.1 [10 points] Find the communicating classes of the Markov chain.
- 4.2 [10 points] Determine if each communicating class is recurrent or transient.

☐ **Problem 5** The Markov chain $X = \{X_n\}_{n=0}^{\infty}$ has state space $\mathbf{X} = \{a, b, c, d, e\}$ and

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- 5.1 [10 points] Find the communicating classes of the Markov chain.
- 5.2 [10 points] Determine if each communicating class is recurrent or transient.

☐ **Bonus** [X points] If the Markov Chain in Problem 5 has initial state $X_0 = c$, what is the probability that X_5 is in the set $\{b, c, d\}$?

Note: looking back at Lecture 8 and at Example 6 in §4.1 of [Ross]*, if one identifies* $\{a, b, c, d, e\} = \{0, 1, 2, 3, 4\}$*, this Markov chain is a probability model for the experiment of gambling* \$1 *at each time step* $n = 0, 1, 2, \ldots$ *– with independent* 50% *chances of winning each time – starting with* \$2 *and hoping to reach* \$4 *overall before ending up bankrupt at* \$0. *This Bonus problem is therefore asking you: what are the chances that the gambler has not reached the target amount of* \$4 *nor gone bankrupt by the* 5*th gamble?*