

MATH 447/647: Probability Models
Fall 2020 Semester
Homework 3: Due Wednesday October 7, 5:30pm ET.

Instructions

- Write your full name, “Homework 3”, and the date at the top of the first page.
- Show all work, including each step of your solution, to earn maximal partial credit.
- Each question has multiple parts. Write legibly and neatly. Box your final answers.
- Use Genius Scan or a similar application to convert your solutions to .pdf format.
- Submit a single .pdf file to Gradescope under the assignment “Homework 3”.
- If you have any questions, email me or come to office hours (WF 11:00am-12:00pm)
- You are encouraged to work together (on Piazza) but must write up your own solutions.

Assignment (4 Problems: $20 + 20 + 20 + 40 = 100$ points total.)

□ **Problem 1** Recall that the *cumulative distribution function* of any random variable X on a probability space (S, P) is the function $F_X : \mathbb{R} \rightarrow [0, 1]$ defined by $F_X(b) = P(X \leq b)$.

- 1.1 [5 points] If X is continuous with probability density function $f_X(x)$, show that

$$f_X(b) = \lim_{h \rightarrow 0} \frac{F_X(b+h) - F_X(b)}{h}.$$

- 1.2 [5 points] Let X be uniformly distributed on $(0, 1)$ as in §2.3.1 [Ross]. Find $F_X(x)$.
- 1.3 [10 points] Let X_1, X_2, X_3 be independent and each uniformly distributed over the interval $(0, 1)$. Find $F_Y(y)$ for $Y = \max(X_1, X_2, X_3)$.

□ **Problem 2** Let X be an exponential random variable with $\lambda = 2$ as in §2.3.2 [Ross].

- 2.1 [10 points] Find $P(X > 9)$.
- 2.2 [10 points] Find $P(X > 10 | X > 1)$.

□ **Problem 3** In Lecture 6, for the triangle Ω with vertices $(0, 0), (2, 0), (0, 2)$ we saw that X and Y defined by the sample outcome $\vec{v} = (X, Y)$ in Ω are dependent random variables according to P_{unif} . X, Y are jointly continuous with joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{2} & \text{if } (X, Y) \in \Omega \\ 0 & \text{if } (X, Y) \notin \Omega. \end{cases}$$

- 3.1 [10 points] Calculate $P(X \geq 1 | Y \leq 1)$.
- 3.2 [10 points] Calculate $\text{Cov}[X, Y]$.

□ **Problem 4** Xander flips a fair coin C_1 and records the outcome as:

$$X = \begin{cases} 1 & \text{if } C_1 = H \\ 0 & \text{if } C_1 = T \end{cases}$$

Consider the two scenarios:

(Scenario 1) Yolanda flips a fair coin C_2 independently of C_1 and writes down:

$$Y = \begin{cases} 1 & \text{if } C_2 = H \\ 0 & \text{if } C_2 = T \end{cases}$$

(Scenario 2) Yolanda does not flip C_2 but instead copies the outcome of Xander's flip of C_1 :

$$Y = \begin{cases} 1 & \text{if } C_1 = H \\ 0 & \text{if } C_1 = T \end{cases}$$

	$Y = 0$	$Y = 1$	
$X = 0$	$p_{X,Y}(0,0)$	$p_{X,Y}(0,1)$	$p_X(0)$
$X = 1$	$p_{X,Y}(1,0)$	$p_{X,Y}(1,1)$	$p_X(1)$
	$p_Y(0)$	$p_Y(1)$	

Recall the *joint probability mass function* of discrete X, Y is

$$p_{X,Y}(x,y) = P(X = x \text{ and } Y = y)$$

and may be represented as a table so $p_X(x)$, $p_Y(y)$ appear as marginal distributions.

- 4.1 [5 points] Find $p_{X,Y}(1,0)$ in Scenario 1.
- 4.2 [5 points] Find $p_{X,Y}(1,0)$ in Scenario 2.
- 4.3 [5 points] Find $p_{X,Y}(x,y)$ in Scenario 1. Write your answer as a table.
- 4.4 [5 points] Find $p_{X,Y}(x,y)$ in Scenario 2. Write your answer as a table.
- 4.5 [5 points] Find $\text{Cov}[X,Y]$ in Scenario 1.
- 4.6 [5 points] Find $\text{Cov}[X,Y]$ in Scenario 2.
- 4.7 [5 points] Find $p_Y(y)$ in Scenario 1.
- 4.8 [5 points] Find $p_Y(y)$ in Scenario 2.

□ **Bonus** [X points] Use conditional probabilities to solve the Monty Hall Problem:

A game show host shows you three doors D_1, D_2, D_3 , tells you that behind one door is a car and behind the other two are goats, and asks you to guess which door the car is behind. You pick D_1 . The host, who knows where the car is, opens up door D_3 and reveals a goat behind D_3 . The host then gives you the option of changing your guess to D_2 . Is it to your advantage to change your guess to D_2 or to stick with your original guess D_1 ?