Name: ________________________________________ Date: January 11, 2021

Probability and Statistics Ph.D. Qualifying Examination. Please show all your work.

1. Xander tosses a fair 6-sided die with outcomes $1, 2, 3, 4, 5, 6$ each equally likely. Let $X$ denote the random outcome of this die roll. As soon as Xander tosses the die, Yolanda calculates $Y = 7 - X$.

   (a) What is the probability that $X \le 3$ and $Y \le 4$?

   (b) Are $X$ and $Y$ independent? Why or why not?

   (c) Are $X$ and $Y$ correlated? Why or why not?

   (d) Which is greater, Var$(X)$ + Var$(Y)$ or Var$(X + Y)$?

   (e) Once the random values of $X$ and $Y$ have been recorded, Bob decides to run a standard Brownian motion $B(t)$ and compare the values of this stochastic process at the random times $t = X$ and $t = Y$. What is the probability that $B(X) < B(Y)$?

2. Let $X_n$, $n = 0, 1, 2, 3, \ldots$ be the simple random walk in the graph below.

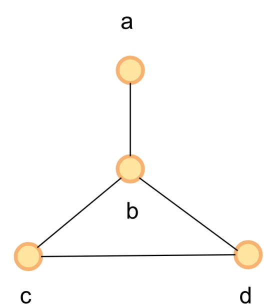


Figure 1: A simple graph on $4$ vertices $\{a, b, c, d\}$ with $4$ edges.

The $1$-step transition probabilities defining $X_n$ are

$$\mathbf{P}_{ij} = \begin{cases} \frac{1}{\deg i} & \text{if } i \to j \text{ connected by an edge} \\ 0 & \text{otherwise} \end{cases}$$

where $\deg(i)$ is the number of edges coming out of the vertex $i$.

   (a) Determine the $1$-step transition matrix $\mathbf{P}$.

   (b) If $X_0 = a$, what is the probability that $X_4 = a$?

   (c) If $X_0$ is chosen uniformly in $\{a, b, c, d\}$, what is the probability that $X_1 = b$?

   (d) Find the equilibrium distribution $\pi$ of $X_n$.

   (e) If $X_0 = c$, what is the probability that $X_n$ returns to $c$ infinitely-many times?

3. Let $N(t)$ be a Poisson process with rate $\lambda = 1$ and arrival times $S_1, S_2, \ldots$.
   (a) State the properties characterizing the Poisson process.
   (b) What is the probability that $S_2 - S_1 < S_3 - S_2$?
   (c) What is the probability that $N(1) = 2$ and $N(3) = 5$?
   (d) What is the probability that $S_1 \leq 1.4$ and $S_2 \leq 2.4$?
   (e) What is $\lim_{t\to\infty} \mathrm{Var}\left(\frac{N(t)}{t}\right)$?

4. Consider the triangular probability distribution with pdf

$$f(y) = \begin{cases} \dfrac{1}{\theta} - \dfrac{|y|}{\theta^2}, & |y| \leq \theta \\ 0, & \text{elsewhere} \end{cases}$$

   where $\theta > 0$ is an unknown parameter. Let $(Y_1, \ldots, Y_n)$ be a random sample from this distribution.

   (a) Calculate the mean $\mu$ and variance $\sigma^2$ of this distribution.
   (b) Find an estimator $\hat{\theta}_1$ of $\theta$ by the method of moments.
   (c) Is $\hat{\theta}_1$ unbiased? Is it consistent?
   (d) Let $\hat{\theta}_2$ be the maximum likelihood estimator of $\theta$. Show that $\hat{\theta}_2 \geq \max_{1\leq i\leq n} |Y_i|$.
   (e) Assuming $\max_{1\leq i\leq n} |Y_i| > 0$, show that $\hat{\theta}_2$ is uniquely determined by

$$\sum_{i=1}^{n} \frac{|Y_i|}{\hat{\theta}_2 - |Y_i|} = n.$$

   That is, show that $x = \hat{\theta}_2$ is the only solution to the equation $\sum_{i=1}^{n} \frac{|Y_i|}{x-|Y_i|} = n$ such that $x \geq \max_{1\leq i\leq n} |Y_i|$.
   (f) Consider the estimator $\hat{\theta}_3 = \frac{c}{n}\sum_{i=1}^{n} |Y_i|$, where $c$ is a real constant. How should $c$ be chosen for $\hat{\theta}_3$ to be unbiased?
   (g) From here onwards, it is assumed that $c$ is such that $\hat{\theta}_3$ is unbiased. Show that $\hat{\theta}_3$ is a consistent estimator of $\theta$.
   (h) Prove that

$$\sqrt{n}\big(\hat{\theta}_3 - \theta\big) \longrightarrow N(0, \theta^2/2) \quad \text{as } n \to \infty,$$

   where $N(m, v)$ is a normal distribution with mean $m$ and variance $v$.
   (i) Deduce an approximate confidence interval of the form $[L, \infty)$ for $\theta$ at the level $1 - \alpha$. That is, give a formula for $L = L(\hat{\theta}_3, n, \alpha)$ where $\alpha \in (0, 1)$. *Hint: first find an asymptotically pivotal quantity for $\theta$.*
   (j) Suppose that you want to test the hypothesis $H_0 : \theta = 1$ versus $H_a : \theta > 1$ at the significance level $\alpha = 0.01$ and observe the following result: $\hat{\theta}_3 = 1.05$ with $n = 50$. Would you retain or reject $H_0$? *Hint: use the asymptotically pivotal quantity of part (i) to conduct the test.*