University of the Witwatersrand, Johannesburg

Course Code No(s)	**STAT4110/ STAT7035**
Course Description name(s)	**Operations Research**
Due date	**May 4, 2020 (Provisional)**
Year of Study	**First Year**
Programme Code (Degree code)	**BSc Hons/MSc**
Faculty/ies presenting	**Science**
Internal examiner(s) and telephone extension number(s)	**Dr HW Chipoyera (76295)**
Open Book Exam	**Yes** √ / **No**
Electronic Devices/Computers. Please indicate if any devices will be used during the exam and if so what device /devices will be used?	**Scientific Calculators**
Special materials required (graph/ music/drawing paper) maps, diagrams, tables, computer cards, etc.	
Time allowance	**Course Nos.** | **STAT4110/ STAT7035** | **Hours** | **
Instructions to candidates (Examiners may wish to use this space to indicate, inter alia, the contribution made by this examination or test towards the year mark, if appropriate)	**Answer ALL questions**, **The answer to each question should start on a new page.** **Use the assignment cover page provided.**

Question 1 (10 marks)

A bank is in the process of devising a loan policy that involves a maximum of 120 million rand. The information in Table 1 provides the pertinent data about available loans.

- Bad debts are unrecoverable and produce no interest revenue.
- Competition with other financial institutions dictates the allocation of at least 40% of the funds to farm and commercial loans.
- To assist the housing industry in the country, home loans must equal at least 50% of the personal, car and home loans.
- The bank limits the overall ratio of bad debts on all loans to at most 4%.

Table 1: **Information on different types of loans**

Type of loan	Interest rate	Bad-debt ratio
Personal	0.140	0.10
Car	0.130	0.07
Home	0.120	0.07
Farm	0.125	0.05
Commercial	0.100	0.02

a) Formulate the Linear program (LP) to help find an optimal solution to the problem. . [5]

b) Use LINDO to determine an optimum solution to the problem. (You must submit the LINDO output in conjunction with a brief explanation of what the output entails.) [5]]

Question 2 (10 marks)

Suppose The Department of Home affairs has summarized the minimum numbers of immigration officers needed at Beit Bridge (BB) Border post over each 4-hour period everyday in March in Table 2

Each immigration officer works a shift that lasts 8 hours and the shifts start at 12 midnight, 4am, 8am, 12 noon, 4pm and 8pm.

Table 2: **Numbers of immigration officers needed at BB in March**

Time period	Minimum number of officers required
12midnight-4am	80
4am-8am	70
8am-12noon	40
12noon-4pm	30
4pm-8pm	40
8pm-12 midnight	90

a) Formulate the Linear program (LP) to help find an optimal solution to the problem. [5]

b) Use LINDO to determine an optimum solution to the problem. (You must submit the LINDO output in conjunction with a brief explanation of what the output entails.) [5]]

Question 3 (10 marks)

A cell-phone manufacturing company must determine how many cellphones should be produced over the next three years. They anticipate that demand in the first year will be 30000 units, in the second year it will be 70000 units and in the third year it will be 40000 units. The company must meet demand in time. At the beginning of the first quarter, they have 10000 units in stock. The company can produce a maximum of 40000 units per annum in regular time at a total cost of R1,000,000 per every batch of 1000 cellphones. With the provision of workers working overtime, the company can produce a batch of 1000 cellphones at a cost of R1,200,000.

At the end of each year (after production has occurred and the current year's demand has been satisfied), a carrying or holding cost of R50,000 per batch of 1000 cellphones is incurred.

a) Formulate the Linear program (LP) to help find an optimal solution to the problem. . [5]

b) Use MATHEMATICA to determine an optimum solution to the problem. (You must submit the LINDO output in conjunction with a brief explanation of what the output entails.) [5]

Question 4 (12 marks)

Mzanzi and Van Rooyen (pty) ltd is considering 5 large capital investments. Each investment can be made only once. The investments differ in the long-run profit they

Table 3: **Profit and capital required data for Mzanzi and Van Rooyen (pty) ltd investment options**

	Investment opportunity				
	A	B	C	D	E
Estimated profit (in millions of rand)	25	15	16	19	12
Capital required (in millions of rand)	37	32	19	33	16

will generate as well as the amount of capital required as shown in table refinvest below:

The total amount of capital for these investments is 90 million rand. The following information relate to the investment rules

- Investment D can only be undertaken if at least one of Investment A and B are undertaken
- Investments A and C are mutually exclusive, i.e. they cannot both be undertaken and in other words an investment of one of them precludes an investment in the other
- An investment in B can only be done if an investment in D has been made and vice versa

Suppose Mzanzi and Van Rooyen (pty) Ltd has come to you for advice and you have decided to model their problem as an integer programming problem whose objective is to select the combination of capital investments that will maximize the total expected profit.

a) By appropriately defining the decision variables as $x_A, \cdots, x_E$, formulate the problem as an integer programming problem as part of a process to finding a solution to Mzanzi and Van Rooyen's problem. [5]

b) Write down a LINDO code that you would use to find the optimal solution to the problem. [4]

c) Print and submit the associated LINDO output accompanied by a short interpretation of what the optimal solution entails. [3]

Question 5 (10 marks)

Consider the following linear programming (LP) problem:

$$\begin{aligned} \text{Minimize} : z &= 4x_1 + 6x_2 \\ \text{subject to} : x_1 + 2x_2 + 3x_3 &\leq 30 \\ 2x_1 + 3x_2 &\geq 6 \\ x_3 &\geq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

In solving the problem using the Big-M simplex algorithm, a slack variable s_1 is augmented to first constraint while surplus variables s_2 and s_3 are augmented to the second and third constraints, respectively; in addition artificial variables, R_2 and R_3 are augmented to the second and third constraint, respectively.

a) Show that Tableau 2 in Table 4 is an optimal tableau of the LP problem. [5]

Table 4: Tableau 2 - optimal tableau

Basic	x_1	x_2	x_3	s_1	s_2	s_3	R_2	R_3	Solution
z	0	0	0	0	-2	0	$-M+2$	$-M$	12
s_1	$-\frac{1}{3}$	0	0	1	$\frac{2}{3}$	3	$-\frac{2}{3}$	-3	14
x_2	$\frac{2}{3}$	1	0	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	2
x_3	0	0	1	0	0	-1	0	1	4

b) On the basis of Tableau 2, eplain why you would say the LP problem has alternative optimal solution(s) . [2]

c) Using the simplex algorithm, show that Tableau 3 in Table 5 is an alternative optimal tableau associated with basis $BV = \{s_1, x_1, x_3\}$. [3]

Table 5: Tableau 3 - optimal tableau

Basic	x_1	x_2	x_3	s_1	s_2	s_3	R_2	R_3	Solution
z	0	0	0	0	-2	0	$-M+2$	$-M$	12
s_1	0	$\frac{1}{2}$	0	1	$\frac{1}{2}$	3	$-\frac{1}{2}$	-3	15
x_1	1	$\frac{3}{2}$	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	3
x_3	0	0	1	0	0	-1	0	1	4

Question 6 (10 marks)

Another optimal solution to the LP in Question 5 has basis $BV = \{s_3, x_1, x_3\}$ and partially completed optimal tableau in Table 6.

a) Copy and complete the tableau. Show all necessary working involving the Revised Simplex Method to complete the table. [7]

Table 6: Tableau 4 - optimal tableau

Basic	x_1	x_2	x_3	s_1	s_2	s_3	R_2	R_3	Solution
z	0	0	0	0	-2	0	$-M+2$	$-M$	12
s_3							$-\frac{1}{6}$	-1	5
x_1							$\frac{1}{2}$	0	3
x_3							$-\frac{1}{6}$	0	9

b) Write down a general expression for the optimal solutions of the LP problem. [3]

Question 7 (8 marks)

Consider the following linear programming (LP) problem in Problem 5 :

$$\begin{aligned} \text{Minimize} : z &= 4x_1 + 6x_2 \\ \text{subject to} : x_1 + 2x_2 + 3x_3 &\le 30 \\ 2x_1 + 3x_2 &\ge 6 \\ x_3 &\ge 4 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

a) Write down the dual of the LP. [4]

b) Write down an optimum solution of the dual LP in a). Show all calculations that involve the optimal solution in Table 5 of Question 5 b). [4]

Question 8 (4 marks)

Consider the following linear programming problem:

$$\begin{aligned} \text{Maximize} \quad z &= 4x_1 - x_2 + 7x_3 + 2x_4 + 5x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + 3x_3 + x_5 \le 6 \\ & 3x_1 + x_2 + 3x_3 + x_4 \le 10 \\ & x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

Use an appropriate version of the complementary slackness theorem to determine whether or not $\mathbf{x} = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)^T = (0, 0, 0, 10, 6)^T$ is an optimal solution of the LP problem?

. [4]

Question 9 (10 marks)

Consider the following linear programming (LP) problem:

$$\begin{aligned}
\text{Minimize} : z &= 4x_1 + 6x_2 \\
\text{subject to} : x_1 + 2x_2 + 3x_3 &\leq 30 \\
2x_1 + 3x_2 &= 6 \\
x_3 &\geq 4 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

a) Use the dual simplex algorithm to find an optimal solution of the LP problem. [5]

b) Use the two-phase simplex algorithm to find an optimal solution of the LP problem.
. [5]

Question 10 (16 marks)

Consider the LP:

$$\begin{aligned}
\text{Maximize} : z &= -5x_1 + 10x_2 \\
\text{subject to} : -4x_1 + 3x_2 &\leq 24 \\
3x_1 + 2x_2 &\geq 5 \\
-x_1 + 6x_2 &\geq 0 \\
x_1, x_2 &\geq 0
\end{aligned}$$

a) Determine the basic feasible solutions of this LP, $\mathbf{b}_i$. [4]]

b) Find a direction of unboundedness, $\mathbf{d}'$ for the LP problem. Stating any result or theorem you use, verify that the direction $\mathbf{d}'$ is indeed a direction of unboundedness.
. [3]

c) Represent point (3, 5) (which lies in the feasible region) as a convex combination of the basic feasible solutions. [3]

d) Demonstrate that the LP is unbounded using simplex tableaus. [3]

e) Find a direction of unboundedness which will help you verify that the LP is unbounded. [3]