Introduction to Optimization
MS&E 111/MS&E 211/ENGR 62 HW1
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Due Date: July 7, 2021, 12:55pm PDT

Please refer to the syllabus for submission instructions, in particular those pertaining to Excel work. Having taken the relevant screenshots, please add them as part of the rest of the write-up in a single pdf file. Please be sure to assign pages in your solution to the relevant problems on Gradescope. The teaching staff will not be responsible for grading unassigned pages.

Problem 1 (10 Points)
A company produces two kinds of products. Product A requires 1/4 hours of assembly labor, 1/8 hours of testing, and costs $1.20 of raw materials. Product B requires 1/3 hours of assembly, 1/3 hours of testing, and costs $0.90 of raw materials. Considering the company's current personnel, there can be at most 90 hours of assembly labor and 80 hours of testing each day. Products A and B have market values of $9 and $8, respectively.

1. Formulate a linear program that can be used to maximize the company's daily profit.
2. Suppose that up to 50 hours of overtime assembly labor can be scheduled, at a cost of $7 per hour. Modify the linear program from Part 1 to incorporate this change. Clearly specify the new LP, not just any changes.

Problem 2 (10 Points)

1. Write the LP

$$\begin{array}{ll} \text{maximize} & 2z_1 + z_2 \\ \text{subject to} & z_1 \geq z_2 \\ & z_1 + z_2 \leq 3 \\ & z_1 \geq 0, \end{array}$$

   with decision variables z_1, z_2 in the vector form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0. \end{array}$$

   Clearly define c, x, A, and b. If you have added or replaced any decision variables, clearly indicate the relationship between the original decision variables and the new ones.

2. Show how $x^2 \leq 16$ can be represented in terms of linear constraints.
3. Show how $|x - 1/2| \leq 1/2$ can be represented in terms of linear constraints.
4. We would now like to show that, on the contrary, it is **not** possible to represent $|x - 1/2| \geq 1/2$ using linear constraints. We'll argue by contradiction.

   In particular, recall that for any LP, any point on the line connecting two feasible points must also be feasible. In other words, the feasible region of any LP must be *convex*.

   Suppose we could write $|x - 1/2| \geq 1/2$ in terms of linear constraints. Then $|x - 1/2| \geq 1/2$ could be used to describe the feasible region of an LP. Show that the feasible region of such an LP cannot be convex. Drawing a diagram of the set $\{x \in \mathbb{R} \mid |x - 1/2| \geq 1/2\}$ might help.

Problem 3: Non-collaborative, no help. (10 Points)
Consider the following linear program:

$$\begin{array}{ll} \text{maximize} & y_1 + 5y_2 \\ \text{subject to} & y_1 + 3y_2 \leq 15 \\ & y_1 - 2y_2 \geq -5 \\ & y_1 + y_2 \geq 4 \\ & y_2 \geq 1 \end{array}$$

1. Plot the feasible region in a two-dimensional graph.

2. Identify all basic feasible solutions.

3. What is the value of the objective function at an optimum solution?

   Explain your answer. You may only use the graph you drew in Part 1 to derive your result; do not use Excel.

   Hint: As mentioned in lecture, if the feasible region of a linear program is non-empty and bounded, then there exists an optimum solution that is a basic feasible solution.

Problem 4 (20 Points)
Please answer each of the following questions to review some important linear algebra concepts.

1. If A and B are matrices of appropriate dimensions, the (i, j)-entry of the product AB is given by

   $$(AB)_{ij} = \sum_k a_{ik} b_{kj}.$$

   Now suppose A and B are $n \times n$ matrices with all entries equal to 1. What is $(AB)_{ij}$?

2. In summation notation, the associative law $(AB)C = A(BC)$ is given by

   $$\sum_j \left(\sum_k a_{ik} b_{kj} \right) c_{jl} = \sum_k a_{ik} \left(\sum_j b_{kj} c_{jl} \right).$$

   With A and B as in Part 1, and assuming C is an $n \times n$ matrix with $c_{il} = 4$, compute $(ABC)_{il}$.

3. *True or False:* All 2 by 2 matrices commute. That is, if A and B are any 2 by 2 matrices, then $AB = BA$. Explain your answer.

4. Let

   $$A = \begin{bmatrix} -1 & 0 & 8 & -2 \\ 4 & 2 & 11 & 6 \\ 2 & 1 & 2 & 3 \end{bmatrix}.$$

   True or False: Suppose we remove one column of A. Regardless of which column we remove, the remaining three columns are linearly independent. Explain your answer.

5. *True or False:*

   (a) If the columns of a matrix A are linearly independent, then $Ax = b$ has exactly one solution for every b.

   (b) The 7 columns of a 4×7 matrix cannot be linearly independent.
   Explain your answers.

6. Consider trying to solve $Ax = b$ where

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 2 & -1 \end{bmatrix}$$

   (a) Find a vector b so that $Ax = b$ has no solution.

   (b) Find a non-zero b so that $Ax = b$ has a solution.

Problem 5 (10 Points)
You may find the data for this problem in the Excel file "HW1_Prob5_sum2021.xlsx" in the Canvas folder for this assignment. In the file, you will find a list of securities, along with their attributes: asset type, price, expected return per dollar invested (calculated using price data), and maximum loss per dollar invested (guaranteed by your broker). Assume that you are given an endowment of \$1000 to invest in these securities, and that you can invest in any fraction of any asset.

1. Write a linear program to allocate your endowment in the securities to maximize the expected return per dollar invested of your portfolio, subject to the following constraints:

   - At least 40% of the portfolio (dollars invested) are in government bonds;
   - No more than 10% of the portfolio (dollars invested) be in alternative investments;
   - The maximum allowable loss of the portfolio is 30%;
   - Due to the recent financial meltdown, you are not allowed to short sell any security (invest a negative amount).

   Solve the linear program using Excel. Note that you may not require all columns provided in the Excel file. Be sure to indicate on your submission if you worked with anyone else.

2. How should the maximum expected return per dollar invested change if the maximum allowed loss is increased?

Problem 6 (10 Points)
The XYZ Corporation is undergoing a radical restructuring and needs to hire consultants to help in several key areas. XYZ has contracts with four consultants (A, B, C, and D). The table below indicates, for each of the consultants, what they charge per hour and how many hours they have available to work for XYZ.

Consultant	rate/hr	total hours available
A	\$300/hr	60
B	\$250/hr	60
C	\$350/hr	50
D	\$200/hr	40

XYZ has four main tasks that need to be completed. The table below indicates how many hours XYZ estimates that it would take each consultant to perform each task. If no amount of time is indicated for the consultant to complete a task, then the consultant cannot perform the task.

Consultant	hrs/task 1	hrs/task 2	hrs/task 3	hrs/task 4
A	30	20		40
B	30	30	30	
C	25	25	25	25
D			40	40

XYZ want to determine how to allocate tasks to consultants at minimum cost. Note that all tasks must be completed, but not all consultants need to be used to their full availability. You may also allow consultants to split tasks. If they do so, you may assume that each consultant completes a constant fraction of the task per unit of time. For example, Consultant A would complete 1/2 of task 1 in 15 hours.

1. Formulate this problem as an LP. Clearly indicate your decision variables, your objective function, and your constraints.
2. Solve your LP using Excel. Be sure to indicate in your submission if you worked with anyone else.