Université de Liège **2020/2021**

Large sample analysis

Examination project

Instructions:

- Please hand in the following **two documents**:
  - A pdf file containing your answers, calculations and a copy of your code.
  - An annotated `R`-file with the executable code.

  Write all code in a single file, clearly indicating which part of the code goes with each of the questions. The answers and calculations can be either handwritten (make sure your handwriting is legible) or typeset using, for example, Overleaf or LaTeX.
- Send both files to `arnout.vanmessem@uliege.be` and carole.baum@uliege.be before **June 14**.
- By June 20, you will receive your scores after which you can decide for an oral continuation of the exam. This oral continuation is optional and will only take place if you specifically request for it; it will contain one theory question and one question related to your project. It can influence your final score with up to 2 marks (both positive or negative). A suitable moment will be sought together.

Exercise 1 (8 marks) *Let Φ be the cumulative distribution function (cdf) of the standard normal distribution. The cdf of a two-parameters Birnbaum-Saunders random variable T is of the form:*

$$F(t \mid \alpha, \beta) := \Phi\left(\frac{\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}}{\alpha}\right), \ t > 0,$$

where $\alpha, \beta > 0$ are the shape and scale parameters, respectively. Such a random variable will be denoted by $T \sim BS(\alpha, \beta)$.

1. *Let $f(t \mid \alpha, \beta)$ be the probability density function (pdf) of T. Calculate $f(t \mid \alpha, \beta)$ and show that the quantile function of T is*

$$Q(p \mid \alpha, \beta) = \frac{\beta}{4}\left[\alpha\Phi^{-1}(p) + \sqrt{4 + (\alpha\Phi^{-1}(p))^2}\right]^2, \ p \in [0, 1].$$

   Plot the functions F, Q and f, for various values of (α, β). What is the shape of the Birnbaum-Saunders distribution when the shape parameter increases to ∞?

2. *Let Z be a random variable defined by*

$$Z := \frac{\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}}{\alpha}.$$

   Show that Z is a standard normal random variable. Give the distribution of the random variable

$$X := \frac{\alpha Z}{2}.$$

Show the relation

$$T = \beta\psi(X) \text{ where } \psi(x) := \left(x + \sqrt{1+x^2}\right)^2.$$

Use this transformation to generate 100 values from $BS(2, 0.5)$ and $BS(2, 1)$ distributions.

3. *Calculate the mean and variance of T.*

4. *Show that if $T \sim BS(\alpha, \beta)$ then $\frac{1}{T} \sim BS\left(\alpha, \frac{1}{\beta}\right)$ and use this result to generate 100 values from $BS(2, 2)$. Calculate the mean and variance of T^{-1}.*

5. *Use a sample of 1000 values from $BS(1.5, 10)$ to illustrate how the maximum likelihood method can be used to estimate the parameters of the Birnbaum-Saunders distribution. Plot the associated log-likelihood function and find the MLE of the parameters. Evaluate the efficiency in term of bias and variance of these MLE through 300 simulations. Illustrate the asymptotic normality of the obtained MLE.*

Exercise 2 (8 marks) *A random variable X follows an inverse Gaussian distribution with parameters $\mu > 0$ and $\lambda > 0$ and we note $X \sim IG(\mu, \lambda)$, if it has the pdf*

$$f(x|\mu, \lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right), \ x > 0.$$

Let X_1 and X_2 two random variables such that

$$X_1 \sim IG(\mu, \lambda) \text{ and } \frac{1}{X_2} \sim IG\left(\frac{1}{\mu}, \frac{\lambda}{\mu^2}\right).$$

Denote by f_1 and f_2 the pdf of X_1 and X_2 respectively. Let Y be the mixture of X_1 and X_2 whose pdf is given by

$$f_Y(y) = \frac{1}{2}f_1(y) + \frac{1}{2}f_2(y).$$

1. *Show that $f_2(x) = \frac{x}{\mu}f_1(x), \ x > 0$ and plot the function f_Y.*

2. *Show that when $\mu = \beta$ and $\lambda = \frac{\beta}{\alpha^2}$, then $Y \sim BS(\alpha, \beta)$.*

3. *Use the Accept-Reject algorithm to simulate a sample of 100 copies of X_2, with instrumental density f_1.*
Hint: In order to be able to use the Accept-Reject algorithm, it is necessary do discretize the support of the densities in the sense that the equality obtained in question 2.1 is valid for $x \in \{x_1, \ldots, x_N\}$, where the x_i are generated following f_1 and N is sufficiently large. Additionally, clearly explain why the Accept-Reject algorithm is not directly applicable in this case.

4. *Generate 100 realizations of the random variable Y.*

Exercise 3 (4 marks) *Read the paper [3] and :*

1. *Summarize the methodology (mainly Section 3.1).*

2. *Redo the simulations in section 4 : generate the model and calculate the MLEs to find the results reported in Tables 1 et 2 (columns MLE).*

References

[1] Birnbaum, Z. W., Saunders, S. C. (1969). *A new family of life distributions*. Journal of Applied Probability, 6 (2), 319–327.

[2] Kundu, D., Balakrishnan, N., Jamalizadeh, A. (2010). *Bivariate Birnbaum-Saunders distribution and associated inference*. Journal of Multivariate Analysis, 101(1), 113–125.

[3] Mazucheli, J., Menezes, A. F., Dey, S. (2018). *The unit-Birnbaum-Saunders distribution with applications.* Chilean Journal of Statistics, 9(1), 47–57.

[4] Pradhan, D., Kundu, D. (2013).*Inference and optimal censoring schemes for progressively censored Birnbaum-Saunders distribution*. Journal of Statistical Planning and Inference, (143) 6, 1098–1108.