Basic Combinatorics (Spring '20)

Instructor: Asaf Shapira

Final Exam

- **Your name: __________________________________**
- **Your ID: __________________________________**
- **Answer all the following 4 questions. Each of the 4 questions has the same weight in your final grade.**
- **Give clear and concise solutions! You will lose points for giving unnecessarily obscure/long/complicated solutions even if they are correct. Furthermore, make sure you justify all your claims, unless you are using claims that were proved in class.**
- **Submit a scanned copy of your solution via MOODLE. This should be in the form of a SINGLE, READABLE PDF file!**

You are not allowed to:

1. **Discuss this exam with any other person.**
2. **Use any material (in printed/electronic/quantum/... state) other than your class notes.**

If two answers will appear as if they were written by the same person I will disqualify both exams!!!

Sign that you agree to follow the above guidelines: ____________

Good Luck!

Problem 1. Prove that the number of different plane partitions that can be placed within an $n \times n \times n$ box is of order $2^{\Theta(n^2)}$. You are **NOT** allowed to use the exact formulas for the number of plane partitions we proved in class. Note that your are required to prove *both* a lower bound (namely, $2^{\Omega(n^2)}$) and an upper bound (namely, $2^{O(n^2)}$).

Clarification: If you do not like the pictorial way in which I defined plane partitions in class (using stacks of boxes that are decreasing in height when going left or right) then here is an equivalent cleaner/clearer/formal one. A plane partition within an $n \times n \times n$ box is just an $n \times n$ matrix A with integer entries, so that for every $0 \le i, j \le n$ we have (i) $0 \le A_{i,j} \le n$ (this is the "height" of the stack of boxes in position i, j), and (ii) $A_{i,j} \ge A_{i,j+1}$ and $A_{i,j} \ge A_{i+1,j}$ (the height of the stack of boxes does not increase when going left/right). So the question asks you to prove upper/lower bounds for the number of such integer matrices.

Solution:

Problem 2. Show that for every $t \geq 2$ we have

$$\sum_{i=0}^{n}(-1)^i\binom{n-i}{i}t^i(t+1)^{n-2i} = \frac{t^{n+1}-1}{t-1}\,.$$

Hint: Color the integers so that if i is not blue then so is $i-1$.

Solution:

Problem 3. In what follows I use the term x-set to denote a set of size x. Let $\mathcal{F}$ be a collection of sets. We say that $\mathcal{F}' \subseteq \mathcal{F}$ is s-nice if the following holds for every s-set S. If there is $F \in \mathcal{F}$ so that $S \cap F = \emptyset$ then there is also $F' \in \mathcal{F}'$ so that $S \cap F' = \emptyset$ (note that S is arbitrary).

Prove that if $\mathcal{F}$ is a collection of r-sets, then there is $\mathcal{F}' \subseteq \mathcal{F}$ which is s-nice and contains at most $\binom{r+s}{s}$ of the r-sets of $\mathcal{F}$.

Hint: Build $\mathcal{F}'$ in the "obvious" way. Use one of the theorems we proved in class in order to prove that this obvious method indeed works.

Solution:

Problem 4. Let A be the $n \times n$ matrix, so that for every $1 \leq i, j \leq n$ the entry $A_{i,j}$ is the Sterling number of the second kind $S_{1+i,j}$. Show that $\det(A) = n!$.

Hint: Define a directed graph whose vertices are the pairs of integers (i, j) with $0 \leq i, j \leq n$ and whose edges are the following: every vertex (i, j) is connected to vertex $(i, j+1)$ with a directed edge of weight 1, and to vertex $(i+1, j)$ with a directed edge of weight j (so the edges along the x-axis have weight 0; equivalently, they can just be removed). For every $1 \leq i \leq n$ let a_i denote the vertex $(n-i, 0)$ and let b_i denote the vertex $(n-i+1, i)$.

Solution: