Basic Combinatorics (Spring '18)

Instructor: Asaf Shapira

Final Exam

- **Your name:** ________________________________
- **Your ID:** ________________________________
- **Answer all the following 4 questions. Each of the 4 questions has the same weight in your final grade.**
- **Give clear and concise solutions! You will lose points for giving unnecessarily obscure/long/complicated solutions even if they are correct. Furthermore, make sure you justify all your claims, unless you are using claims that were proved in class.**
- **Please use the page where the question appears in order to write your solution.**
- **Submit a hard copy of your solution to Michal Amir's mailbox (number 220) . This should be in the form of a SINGLE, READABLE, STAPLED file!**

You are not allowed to:

1. **Discuss this exam with any other person.**
2. **Use any material (in printed/electronic/quantum/... state) other than your class notes.**

If two answers will appear as if they were written by the same person I will disqualify both exams!!!

Sign that you agree to follow the above guidelines: ___________

Good Luck!

Problem 1. Harry is not sure if he wants to stay in his boarding school or go back home. To help him make up is mind, he wants to use his magic coin. When flipped, the magic coin comes up "heads" with probability 0.05 and "tails" with probability 0.95.

Harry decided to do the following; flip the magic coin 100 times and count the number of times it comes up "heads". If this number is even, he will stay at the boarding school, and if this number is odd he will quit school and go back home.

What is the probability that Harry will go back home?

Clarification: Needless to say that your answer should be a "closed form" expression, that is, without any long summations/multiplications.

Solution:

Problem 2. Suppose n is odd. Let $\mathcal{P}([n])$ denote the power set of $[n]$, that is, the 2^n subsets of $[n]$. We say that a family of sets $\mathcal{F} \subseteq \mathcal{P}([n])$ is *nice* if $\mathcal{F}$ does not contain 5 sets A, B, C, D, E satisfying $A \subseteq B \subseteq C \subseteq D \subseteq E$. Show that if $\mathcal{F} \subseteq \mathcal{P}([n])$ is nice then

$$|\mathcal{F}| \leq 2\left(\binom{n}{(n-1)/2} + \binom{n}{(n-3)/2}\right) .$$

In other words, a nice $\mathcal{F}$ contains at most $2\left(\binom{n}{(n-1)/2} + \binom{n}{(n-3)/2}\right)$ of the subsets of $[n]$. Also, find a nice $\mathcal{F} \subseteq \mathcal{P}([n])$ containing $2\left(\binom{n}{(n-1)/2} + \binom{n}{(n-3)/2}\right)$ sets.

Solution:

Problem 3. Prove that if P is a set of 2018 **distinct** points in $\mathbb{R}^2$ then P determines at least 3 distinct (non-zero) distances. That is, there must be 3 pairs of points $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ (with $a_1 \neq b_1$, $a_2 \neq b_2$, $a_3 \neq b_3$) so that $d(a_1, b_1), d(a_2, b_2), d(a_3, b_3)$ are three distinct real numbers (where $d(x, y)$ is the usual Euclidian distance in $\mathbb{R}^2$).

Solution:

Problem 4. Let $p_1(n)$ denote the number of ways we can express n as the sum of positive integers that are not divisible by 2018. Let $p_2(n)$ denote the number of ways we can express n as the sum of positive integers so that each integer appears at most 2017 times. Show that for every n we have $p_1(n) = p_2(n)$.

Clarification: Needless to say that (as usual) order does not matter here. For example, $p_1(3) = 2$, since $\{1, 2\}$ is counted only once (rather than twice as $(1, 2)$ $(2, 1)$).

Solution: