Question 1 (20%)

A random sample of n insurance policies of a particular type, all of which have been in force for 10 years, has been taken. The numbers of claims made are $x_1, x_2, \ldots, x_n$. A Poisson distribution with parameter λ is to be used to model the number of claims.

(i) Derive the maximum likelihood estimator of λ.

(5)

(ii) An insurance company has taken a random sample of 75 policies of the same type. All of these have been in force for ten years. The numbers of claims made are summarised below.

Number of claims	*Number of policies*
0	18
1	25
2	13
3	10
4	6
5	3
Total	75

(a) Use an appropriate statistical test to assess whether a Poisson distribution is an appropriate model for this set of data.

(10)

(b) Obtain an approximate 95% confidence interval for the mean of the underlying distribution.

(5)

Question 2 (20%)

The Weibull distribution has two parameters $a > 0$ and $b > 0$ and has cumulative distribution function (cdf)

$$F(x) = 1 - \exp\left\{-\left(\frac{x}{a}\right)^b\right\}, \quad x > 0.$$

(i) Show that the probability density function is

$$f(x) = \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} \exp\left\{-\left(\frac{x}{a}\right)^b\right\}, \quad x > 0.$$

(4)

(ii) Taking the value of b to be fixed, show that the maximum likelihood estimate for a, based on a random sample of observations $x_1, x_2, \ldots, x_n$ from a Weibull distribution, is given by

$$\hat{a} = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\,b}\right)^{1/b}.$$

(8)

(iii) From past experience it is known that the lives of ball bearings, measured in millions of revolutions, follow a Weibull distribution with $a = 75$ and $b = 3$. However, after a change in production process, it is thought that b should remain unaltered but that the value of a might have changed.

A random sample of 12 ball bearings was tested to failure and the lifetimes at failure, in millions of revolutions, are given below.

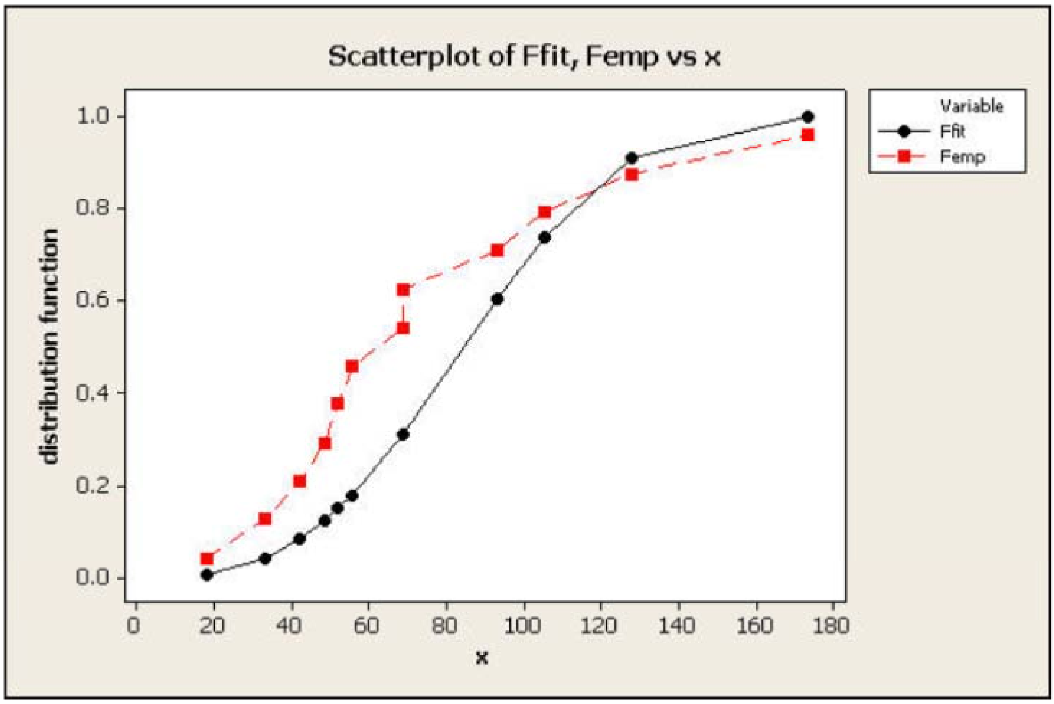
17.88	33.00	42.12	48.48	51.96	55.56	68.64	68.88	93.12
105.12	127.92	173.40						

Calculate the maximum likelihood estimate of a, given that $\sum_{i=1}^{12} x_i^{\,3} = 10\,468\,316$.

(2)

(iv) The plot below shows the empirical cdf (Femp) and the fitted cdf (Ffit), using $b = 3$ and the maximum likelihood estimate for a. Comment on the fit of the proposed model. Suggest two possible courses of action to obtain a model with improved fit.

(6)



Scatterplot of Ffit, Femp vs x
distribution function
x
Variable
Ffit
Femp
0.0
0.2
0.4
0.6
0.8
1.0
0
20
40
60
80
100
120
140
160
180

Question 3 (20%)

The continuous random variables X and Y have joint probability density function $f(x, y) = kxy$ if $0 < x < y < 1$, with $f(x, y) = 0$ elsewhere, where k is a constant.

(i) Evaluate k, and find the marginal probability densities of X and Y. Say, with a reason, whether or not X and Y are independent.

(10)

(ii) Show that, for all non-negative integers r and s, $E(X^r Y^s) = \dfrac{8}{(r+2)(r+s+4)}$.

Hence find the correlation between X and Y.

(10)

Question 4 (20%)

(i) Suppose A and B are independent events, and $\overline{A}, \overline{B}$ denote the complementary events to A, B. Show that $\overline{A}$ and B are independent; deduce that $\overline{A}$ and $\overline{B}$ are independent.

(4)

(ii) The point (X, Y) has the uniform distribution over the unit square $\{0 \le x \le 1,\ 0 \le y \le 1\}$, and u is a given value with $0 < u < 0.5$. Events A and B are defined as $A : \{X \ge 2Y\}$ and $B : \{Y \le X \le Y + u\}$.

(a) Sketch three separate diagrams of the unit square, illustrating the events A, B and $A \cap B$ respectively. (4)

(b) Find the probabilities of these three events, and show that A and B are independent if, and only if, $u = 2/5$. (10)

(c) Deduce the value of $P(\overline{A} \cap \overline{B})$ when $u = 2/5$. (2)

Question 5 (20%)

Some measurements of the volume v (cc) of a casting at seven different temperatures t (in degrees Celsius) of the casting are shown below.

Temperature	18	19	20	21	22	23	24
Volume	10.10	10.80	11.45	12.18	12.80	13.35	15.90

$\Sigma t^2 = 3115$ $\Sigma v^2 = 1092.9774$ $\Sigma tv = 1842.03$

Two linear regressions have been estimated, one using all seven pairs of measurements, and one omitting the last measurement listed (temperature = 24°C). Edited results are shown below; some quantities relating to the first regression have been omitted deliberately.

Regression Analysis: volume versus temperature – all measurements

```
The regression equation is volume = ** + ** temperature

Predictor        Coef     SE Coef       T        P
Constant          **      2.386       -2.31    0.069
temperature       **      0.1131       7.53    0.001

S = 0.5986       R-Sq = **%
```

Regression Analysis: volume versus temperature – omitting measurement at 24°C

```
The regression equation is volume = -1.68 + 0.657 temperature

Predictor        Coef       SE Coef        T        P
Constant        -1.6797     0.2803       -5.99    0.004
temperature      0.65657    0.01362      48.19    0.000

S = 0.05700      R-Sq = 99.8%
```

(i) Plot the data and comment.

(4)

(ii) Calculate the missing estimated coefficients in the first regression, and find the coefficient of determination R^2.

(5)

(iii) Which regression do you think is better, and why?

(4)

(iv) For your preferred regression, interpret its estimated coefficients and its R^2. Explain the meanings of the values of "SE Coef", "T" and "P" given above for it. Why are these values useful?

(7)

Question 7 (20%)

Using Laplace transform, solve the solution $u(x,t)$ for the problem of finding the displacement of an elastic string which is driven by an external force and is determined from

$$u_{xx} + sin\pi x\ sin\omega t = u_{tt}, \qquad 0 < x < 1, \qquad t > 0$$

$$u(0,t) = 0, \qquad u(1,t) = 0, \qquad t > 0$$

$$u(x,0) = 0, \qquad u_t(x,0) = 0, \qquad 0 < x < 1$$

Question 8 (20%)

Find the solution of the heat conduction problem

$$
\begin{aligned}
&100u_{xx} = u_t, \quad 0 < x < 1, \quad t > 0 \\
&u(0,t) = 30, \;\; u(1,t) = 0, \quad t > 0 \\
&u(x,0) = sin2\pi x - sin3\pi x, \;\; 0 \le x \le 1
\end{aligned}
$$

Question 9 (20%)

(a) The relationships between polar coordinates in the plane and rectangular coordinates are given by

$x = r\cos\theta\,, y = r\sin\theta\,,\ r^2 = x^2 + y^2$ and $\tan\theta = y/x$

Determine functions $A(\theta),\ B(\theta),\ C(r,\theta)$ and $D(r,\theta)$ such that *u(x,y)* can be expressed as

$$\frac{\partial u}{\partial x} = A(\theta)\frac{\partial u}{\partial r} + C(r,\theta)\frac{\partial u}{\partial \theta}$$
$$\frac{\partial u}{\partial y} = B(\theta)\frac{\partial u}{\partial r} + D(r,\theta)\frac{\partial u}{\partial \theta}$$

(6)

(b) Determine functions *R(r)* and *S(r)* such that the two-dimensional Laplacian of *u(x,y)* can further be written as

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + R(r)\frac{\partial u}{\partial r} + S(r)\frac{\partial^2 u}{\partial \theta^2}$$

(6)

(c) Determine the groundwater depth $h(r,\theta)$ which satisfies the Laplace's equation for the two-dimensional steady groundwater flow and the boundary conditions.

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2} = 0$$
$$h(a,\theta) = h_1, \quad \frac{\partial h}{\partial r}(b,\theta) = \frac{h_2}{b-a}, \quad 0 \le \theta < 2\pi, \quad 0 < a < b$$

(8)

Question 10 (20%)

Consider the transverse displacement of a vibrating beam of length L whose ends are in simply-supported conditions. The beam is set in motion with initial velocity g(x) and initial displacement given by f(x), $0 \le x \le L$. The transverse displacement $u(x,t)$ can satisfy the following governing equation, initial and boundary conditions

$$a^2 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0, \quad 0 < x < L, \quad t > 0 \qquad where \;\; a^2 = \frac{EI}{\rho A}$$
$$u(0,t) = 0, \;\; u(L,t) = 0, \quad t > 0$$
$$u_{xx}(0,t) = 0, \;\; u_{xx}(L,t) = 0, \quad t > 0$$
$$u(x,0) = f(x), \;\; u_t(x,0) = g(x), \quad 0 \le x \le L$$

where EI is the bending stiffness, ρ is the density and A is the cross-sectional area of the beam.

(a) Using the method of separation of variables (i.e. let $u(x,t) = X(x)T(t)$), derive the following ordinary differential equations

$$\frac{d^4X}{dx^4} - \alpha^4 X = 0$$
$$\frac{d^2T}{dt^2} + a^2\alpha^4 T = 0$$

where α is a positive real constant. (2)

(b) Determine the general solutions of the ordinary differential equations (4)

(c) Determine the fundamental solutions of the partial differential equation and boundary conditions. (10)

(d) Determine a formal series expansion for the transverse displacement $u(x,t)$ that also satisfies the initial conditions. (4)