

Examination length: TWENTY-FOUR (24) hours

Note: Although the window for completion is TWENTY-FOUR (24) hours, this exam paper is designed to be completed in TWO (2) hours.

Question 1

- a) Consider the linear regression model with a single regressor x_t and an error term u_t :

$$y_t = \beta x_t + u_t \quad (1)$$

x_t and u_t are both mean zero, stationary processes. Data (y_t, x_t) , $t = 1, \dots, T$ is observed. Assume T is large. It is known that x_t and u_t follow a periodic cycle. For 10 observations, from $t = 1$, $E(u_t) = 1$ and $E(x_t) = -1$, for the next 10 observations, from $t = 11$, $E(u_t) = -1$ and $E(x_t) = 1$ and so on. Apart from this cycle, x_t and u_t are independent.

- i) [5 points] Will estimating equation (1) yield a consistent estimate of β ? Explain your answer.
- ii) [8 points] Imagine that you define a periodic dummy variable, z_t , which takes a value of 1 for the first ten periods and -1 for the next ten periods and so on. If you include z_t as an additional regressor in equation (1), would your answer to the previous question change?

- b) Consider the following ARMA(1,1) model:

$$y_t = 0.8y_{t-1} + \varepsilon_t + 0.5\varepsilon_{t-1}. \quad (2)$$

where $\varepsilon_t \sim N(0, 1)$ and is serially independent..

- i) [4 points] Explain why the model in equation (2) is stationary and invertible.
- ii) [6 points] Derive the autocovariance function of the model in equation (2).
- iii) [7 points] Derive an expression for the j th coefficient in the $MA(\infty)$ version of the model in equation (2).

Question 2

Consider the following ARCH model:

$$\begin{aligned} r_t &= \mu + \phi h_t + \eta_t; \\ \eta_t &= h_t \varepsilon_t; \quad \varepsilon_t \sim iid N(0, 1) \\ h_t^2 &= \alpha_0 + \alpha_1 \eta_{t-1}^2 + \alpha_2 \eta_{t-2}^2 + \alpha_3 \eta_{t-3}^2 + \beta_1 h_{t-1}^2 \end{aligned}$$

where the coefficients are such that stationarity of η_t and non-negativity of h_t^2 are satisfied. Furthermore, assume that the fourth moment $E[\eta_t^4]$ exists and does not depend on time.

- (a) [5 points] Compute the conditional mean and the conditional variance of r_t .
- (b) [5 points] Show that η_t^2 follows an ARMA process.
- (d) [5 points] Show that η_t features excess kurtosis, i.e.

$$\kappa = \frac{E[\eta_t^4]}{E[\eta_t^2]^2} > 3.$$

Which stylized fact in the data is consistent with this finding?

- (e) [10 points] Set $\beta_1 = \alpha_2 = \alpha_3 = \phi = 0$.
- i Derive an expression for the fourth moment $E[\eta_t^4]$ as a function of α_1 .
 - ii Show that $\alpha_1^2 < 1/3$ is a necessary condition for the existence of the fourth moment.
- (f) [10 points] Set $\beta_1 = \alpha_2 = \alpha_3 = \phi = 0$ and $\alpha_1^2 < 1/3$.
- i Derive the kurtosis of η_t as a function of α_1 .
 - ii Show that the kurtosis of η_t exceeds the kurtosis of ε_t by the quantity $6\alpha_1^2/\{1 - 3\alpha_1^2\}$.

Question 3

Define $f_{t+\tau|t}$ as the forecast of the variable $Y_{t+\tau}$ at the τ -step-ahead horizon, based on the information set Ω_t . An economist wishes to produce optimal forecasts for a quadratic loss function and evaluate their optimality.

- (a) [5 points] Suppose $Y_t = 0.5Y_{t-1} + 0.1Y_{t-2} + \varepsilon_t + 0.5\varepsilon_{t-1} + 0.25\varepsilon_{t-2}$. Derive the optimal one-step ahead forecast and corresponding forecast error for a quadratic loss.
- (b) [5 points] Suppose $Y_t = 0.5Y_{t-1} + 0.1Y_{t-2} + \varepsilon_t + 0.5\varepsilon_{t-1} + 0.25\varepsilon_{t-2}$. Derive the optimal τ -step-ahead forecast and corresponding forecast error for a quadratic loss, for $\tau \geq 2$.

- (c) Consider the regression

$$e_{t+\tau|t} = \alpha + \eta_t, \quad (3)$$

where $e_{t+\tau|t} = Y_{t+\tau} - f_{t+\tau|t}$ are the τ -step-ahead forecast errors and η_t is an error term.

- (i) [5 points] What is the implication of $f_{t+\tau|t}$ being the best forecast (for a quadratic loss function) on the coefficient α in the regression (3)?
- (ii) [5 points] How would you test for this implication?
- (iii) [5 points] Which estimator for the variance of $\hat{\alpha}$ should you use? Why?
- (d) [5 points] Now consider the following *alternative* loss function: $L(e_{t+\tau|t}) = \exp(e_{t+\tau|t}) - e_{t+\tau|t} - 1$. Explain how you would modify equation (3) to test forecast optimality for this alternative loss function.
- (e) [5 points] Consider the task of producing a time series of realized forecast errors $e_{t+\tau|t}$ to use in equation (3). Describe in detail how this can be achieved in practice. Describe and discuss the advantages and disadvantages of a recursive versus a rolling estimation scheme for such exercise.