Solution of 2016 Midterm (Math 2033)

① We have $A\subseteq[3,6]$ and $x\in[5,\sqrt{80})$. So $3\le y\le 6$ and $5\le x<\sqrt{80}$
Then $9\le y^2\le 36$ and $9\le x^2-16<64$ ($\Leftrightarrow 3\le\sqrt{x^2-16}<8$). So
$12\le y^2+\sqrt{x^2-16}<44$. Hence B is bounded above by 44 and below by 12.
By supremum limit theorem, $\exists\, y_n\in A$ such that $\lim\limits_{n\to\infty} y_n=6$.
By infimum limit theorem, $\exists\, y_n'\in A$ such that $\lim\limits_{n\to\infty} y_n'=3$.
Next let $x_n=\sqrt{80}-\frac{1}{n}$ and $x_n'=5+\frac{1}{n\sqrt2}$, then $x_n, x_n'\in[5,\sqrt{80})\setminus\mathbb{Q}$
and $\lim\limits_{n\to\infty}x_n=\sqrt{80}$, $\lim\limits_{n\to\infty}x_n'=5$. Then $y_n^2+\sqrt{x_n^2-16}\in B$ and $\lim\limits_{n\to\infty} y_n^2+\sqrt{x_n^2-16}=44$.
Also $y_n'^2+\sqrt{x_n'^2-16}\in B$ and $\lim\limits_{n\to\infty} y_n'^2+\sqrt{x_n'^2-16}=12$. $\therefore$ $\sup B=44$ and $\inf B=12$.

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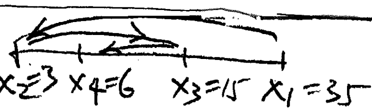
② Sketch: $x_1=35$, $x_2=3$, $x_3=15$, $x_4=6$



$x=\frac{120}{5+x} \Leftrightarrow x^2+5x-120=0 \Leftrightarrow x=\frac{-5\pm\sqrt{25+480}}{2}=\frac{-5\pm\sqrt{505}}{2}$ (reject −)

Solution Let $I_n=[x_{2n},x_{2n-1}]$ for $n=1,2,3,\ldots$
Claim: $I_n\supseteq I_{n+1}$ ($\Leftrightarrow x_{2n}\le x_{2n+2}\le x_{2n+1}\le x_{2n-1}$) for $n=1,2,3,\ldots$
Case $n=1$, $x_2=3\le x_4=6\le x_3=15\le x_1=35$
Suppose case n is true. Then $x_{2n}\le x_{2n+2}\le x_{2n+1}\le x_{2n-1}$. We have
$5+x_{2n}\le 5+x_{2n+2}\le 5+x_{2n+1}\le 5+x_{2n-1}\Rightarrow \frac{1}{5+x_{2n}}\ge\frac{1}{5+x_{2n+2}}\ge\frac{1}{5+x_{2n+1}}\ge\frac{1}{5+x_{2n-1}}$
$\Rightarrow x_{2n+1}=\frac{120}{5+x_{2n}}\ge x_{2n+3}=\frac{120}{5+x_{2n+2}}\ge x_{2n+2}=\frac{120}{5+x_{2n+1}}\ge x_{2n}=\frac{1}{5+x_{2n-1}}$
$\Rightarrow 5+x_{2n+1}\ge 5+x_{2n+3}\ge 5+x_{2n+2}\ge 5+x_{2n}\Rightarrow \frac{1}{5+x_{2n+1}}\le\frac{1}{5+x_{2n+3}}\le\frac{1}{5+x_{2n+2}}\le\frac{1}{5+x_{2n}}$
$\Rightarrow x_{2n+2}=\frac{120}{5+x_{2n+1}}\le x_{2n+4}=\frac{120}{5+x_{2n+3}}\le x_{2n+3}=\frac{120}{5+x_{2n+2}}\le x_{2n+1}=\frac{120}{5+x_{2n}}$.
This completes M.I. So the claim is true.
By nested interval theorem, $\lim\limits_{n\to\infty}x_{2n}=a$ and $\lim\limits_{n\to\infty}x_{2n-1}=b$. Now $\lim\limits_{n\to\infty}x_{2n+1}=\lim\limits_{n\to\infty}\frac{120}{5+x_{2n}}$
$\Rightarrow b=\frac{120}{5+a}$ and $\lim\limits_{n\to\infty}x_{2n}=\lim\limits_{n\to\infty}\frac{120}{5+x_{2n-1}}\Rightarrow a=\frac{120}{5+b}$. We have $5b+ab=120=5a+ab$.
This implies $a=b$. Then $a=\lim\limits_{n\to\infty}x_{n+1}=\lim\limits_{n\to\infty}\frac{120}{5+x_n}=\frac{120}{5+a}\Rightarrow a^2+5a-120=0$
$\Rightarrow a=\frac{-5\pm\sqrt{25+480}}{2}=\frac{-5\pm\sqrt{505}}{2}$. Since $x_n\ge 3$, $a=\lim\limits_{n\to\infty}x_n$ is $\frac{-5+\sqrt{505}}{2}$.

③ $S=\{r\sqrt{2} : r\in\mathbb{Q}\} = \bigcup_{\underbrace{r\in\mathbb{Q}}_{\text{countable}}} \underbrace{\{r\sqrt{2}\}}_{1\text{ element} \Rightarrow \text{countable}}$ is countable by countable union theorem.

Let $W=\{2^x\sin y - 2^y\cos x : x,y\in S\}$. Then $W=\bigcup_{\underbrace{(x,y)\in S\times S}_{\text{countable}}} \underbrace{\{2^x\sin y - 2^y\cos x\}}_{1\text{ element} \Rightarrow \text{countable}}$

is countable. Then $\underbrace{\mathbb{R}}_{\text{uncountable}}\setminus \underbrace{W}_{\text{countable}}$ is uncountable. So $\mathbb{R}\setminus W$ is an infinite set.

Therefore, there exist infinitely many real numbers c such that $2^x\sin y - 2^y\cos x = c$ has no solution with $x,y\in S$.

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④ Sketch $\frac{4n\sqrt{n}-3}{n\sqrt{n}+5}\to 4$, $\frac{6n^2-2n+3}{3n^2-1}\to 2$

$$\begin{cases} \left|\frac{4n\sqrt{n}-3}{n\sqrt{n}+5}-4\right| = \frac{|-23|}{n\sqrt{n}+5} \leq \frac{23}{n\sqrt{n}} = \frac{23}{n^{3/2}} < \frac{\varepsilon}{2} \Leftrightarrow \left(\frac{46}{\varepsilon}\right)^{2/3} < n \\ \left|\frac{6n^2-n+3}{3n^2-1}-2\right| = \frac{|-n+5|}{3n^2-1} \leq \frac{n+5}{3n^2-n^2} \leq \frac{6n}{2n^2} = \frac{3}{n} < \frac{\varepsilon}{2} \Leftrightarrow \frac{6}{\varepsilon} < n \end{cases}$$

Solution For every $\varepsilon>0$, by Archimedean Principle, $\exists K\in\mathbb{N}$ such that $K>\max\left\{\left(\frac{46}{\varepsilon}\right)^{2/3}, \frac{6}{\varepsilon}\right\}$ (or take $K=\left\lceil \max\left\{\left(\frac{46}{\varepsilon}\right)^{2/3}, \frac{6}{\varepsilon}\right\}\right\rceil$)

Then $n\geq K \Rightarrow \left|\frac{4n\sqrt{n}-3}{n\sqrt{n}+5}+\frac{6n^2-n+3}{3n^2-1}-6\right| \leq \left|\frac{4n\sqrt{n}-3}{n\sqrt{n}+5}-4\right| + \left|\frac{6n^2-n+3}{3n^2-1}-2\right|$

$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$.