Applied Physics 158

Numerical Linear Algebra

1^{st} sem, A.Y. 2020–2021

1 Methods

Linear systems can generally be written in the form

$$Ax = b \tag{1}$$

where A is an $m \times n$ matrix, x is a vector of length n, and b is a vector of length m. There can be no solution, one solution, or infinitely many solutions for such configuration. In matrix form this can be written as

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m-1,1} & a_{m-1,2} & \cdots & a_{m-1,n-1} & a_{m-1,n} \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n-1} & a_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{m-1} \\ b_m \end{pmatrix} \tag{2}$$

which corresponds to the linear system

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n-1}x_{n-1} + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n-1}x_{n-1} + a_{2,n}x_n &= b_2 \\ &\vdots \\ a_{m-1,1}x_1 + a_{m-1,2}x_2 + \cdots + a_{m-1,n-1}x_{n-1} + a_{m-1,n}x_n &= b_{m-1} \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n-1}x_{n-1} + a_{m,n}x_n &= b_m \end{aligned} \tag{3}$$

1.1 Gaussian Elimination

For simplicity, let A be a square matrix. The premise of Gaussian elimination is to determine from the augmented matrix

$$\left(\begin{array}{ccccc|c} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & a_{2,n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} & b_{n-1} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & a_{n,n} & b_n \end{array}\right) \tag{4}$$

a row echelon form that can easily give the solutions of the system of equations. This is shown by

$$\left(\begin{array}{ccccc|c} a'_{1,1} & a'_{1,2} & \cdots & a'_{1,n-1} & a'_{1,n} & b'_1 \\ 0 & a'_{2,2} & \cdots & a'_{2,n-1} & a'_{2,n} & b'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a'_{n-1,n-1} & a'_{n-1,n} & b'_{n-1} \\ 0 & 0 & \cdots & 0 & a'_{n,n} & b'_n \end{array}\right) \tag{5}$$

and can be calculated by following the rules of solving linear equations:

- a nonzero constant may be multiplied on both sides of any of the equation,
- two rows can be switched with their corresponding b values, and
- adding and/or subtracting two rows are allowed and placing it on one of those rows.

An algorithmic way to solve this is by looping the successive assignment

$$a_{i,j} = a_{i,j} - \frac{a_{i,k}}{a_{k,k}} a_{k,j} \tag{6}$$

for each step k from $k = 1, 2, \ldots, n-1$ for values $i = k+1, \ldots, n$ and $j = k+1, \ldots, n+1$.

This can be decomposed further into a reduced row echelon form

$$\left(\begin{array}{ccccc|c} a''_{1,1} & 0 & \cdots & 0 & 0 & b''_1 \\ 0 & a''_{2,2} & \cdots & 0 & 0 & b''_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a''_{n-1,n-1} & 0 & b''_{n-1} \\ 0 & 0 & \cdots & 0 & a''_{n,n} & b''_n \end{array}\right) \tag{7}$$

which creates a diagonal matrix that easily gives away the value of each x element. This is a variant of the Gaussian elimination called the Gauss-Jordan elimination. Algorithmically, this is given by

$$a_{i,j} = \begin{cases} a_{i,n+1} - \frac{a_{i,k}}{a_{k,k}} a_{k,n+1} & i = 1, 2, \ldots, k-1, k+1, \ldots, n \\ a_{i,n+1} & i = k \end{cases} \tag{8}$$

for each step k from $k = 1, 2, \ldots, n-1$.

1.2 LU Decomposition

Suppose the square matrix A can be broken down into two factors L and U such that

$$A = LU \tag{9}$$

and the decomposition is composed of a lower triangular matrix L and an upper triangular matrix U. This technique is called an LU decomposition or the Doolittle's method. The matrix multiplication LU can be written as

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ l_{2,1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ l_{n-1,1} & l_{n-1,2} & \cdots & 1 & 0 \\ l_{n,1} & l_{n,2} & \cdots & l_{n,n-1} & 1 \end{pmatrix} \begin{pmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,n-1} & u_{1,n} \\ 0 & u_{2,2} & \cdots & u_{2,n-1} & u_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & u_{n-1,n-1} & u_{n-1,n} \\ 0 & 0 & \cdots & 0 & u_{n,n} \end{pmatrix} \tag{10}$$

This can be simplified into the summation notation

$$a_{i,j} = \sum_{k=1}^{n} l_{i,k} uk, j \tag{11}$$

As an algorithm, the values can be determined by the row assignment for r in $r = 1, 2, \ldots, n$ of U

$$u_{r,j} = a_{r,j} - \sum_{k=1}^{r-1} l_{r,k} u_{k,j} \tag{12}$$

for $j = r, r+1, \ldots, n$. The columns of L can immediately be assigned for r in $r = 1, 2, \ldots, n-1$

$$l_{i,r} = \frac{a_{i,r} - \sum_{k=1}^{r-1} l_{i,k} u_{k,r}}{u_{r,r}} \tag{13}$$

for $i = r+1, r+2, \ldots, n$.

1.3 Cholesky and LDL Decomposition

Suppose the square matrix A can be broken down into two factors L and L^T such that

$$A = LL^T \tag{14}$$

and the decomposition is composed of a lower triangular matrix L and its transpose L^T This technique is called the Cholesky Decomposition. The matrix multiplication LL^T can be written as

$$\begin{pmatrix} l_{1,1} & 0 & \cdots & 0 & 0 \\ l_{2,1} & l_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ l_{n-1,1} & l_{n-1,2} & \cdots & l_{n-1,n-1} & 0 \\ l_{n,1} & l_{n,2} & \cdots & l_{n,n-1} & l_{n,n} \end{pmatrix} \begin{pmatrix} l_{1,1} & l_{2,1} & \cdots & ln-1,1 & l_{n,1} \\ 0 & l_{2,2} & \cdots & l_{n-1,2} & l_{n,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & l_{n-1,n-1} & l_{n,n-1} \\ 0 & 0 & \cdots & 0 & l_{n,n} \end{pmatrix} \tag{15}$$

By evaluating this matrix multiplication and comparing the elements to A, an algorithm can be created to determine the values of L.

Generally, the values on the diagonal of L can be determined by

$$l_{k,k} = \sqrt{a_{k,k} - \sum_{j=1}^{k-1} l_{k,j}^2} \tag{16}$$

for k in $k = 1, 2, \ldots, j$ and for the remaining elements below the diagonal of L,

$$l_{i,k} = \frac{1}{l_{k,k}} \left(a_{i,k} - \sum_{j=1}^{k-1} l_{i,j} l_{k,j} \right) \tag{17}$$

for k in $k = 1, 2, \ldots, j$, for j in $j = 1, 2, \ldots, i$, and for i in $i = 1, 2, \ldots, n$.

Related to this is the LDL decomposition. Suppose the square matrix A can be broken down into three factors L, D, and L^T such that

$$A = LDL^T \tag{18}$$

and the decomposition is composed of a lower triangular matrix L, its transpose L^T, and a diagonal matrix D. The matrix multiplication LDL^T can be written as

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ l_{2,1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ l_{n-1,1} & l_{n-1,2} & \cdots & 1 & 0 \\ l_{n,1} & l_{n,2} & \cdots & l_{n,n-1} & 1 \end{pmatrix} \begin{pmatrix} d_{1,1} & 0 & \cdots & 0 & 0 \\ 0 & d_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{n-1,n-1} & 0 \\ 0 & 0 & \cdots & 0 & d_{n,n} \end{pmatrix} \begin{pmatrix} 1 & l_{2,1} & \cdots & ln-1,1 & l_{n,1} \\ 0 & 1 & \cdots & l_{n-1,2} & l_{n,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & l_{n,n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{19}$$

Similar to Cholesky decomposition, by evaluating this matrix multiplication and comparing the elements to A, an algorithm can be created to determine the values of L and D.

1.4 Jacobi Method

Suppose the square matrix A can be broken down into three addends L, D, and U such that

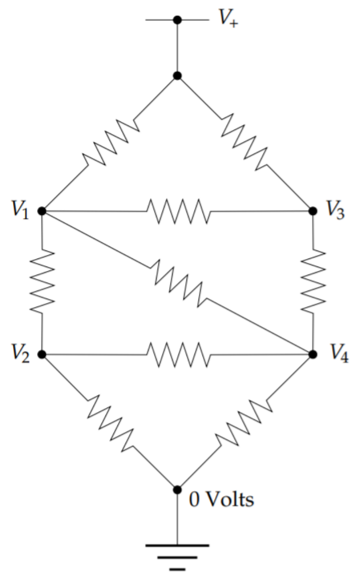
$$A = L + D + U \tag{20}$$

and the decomposition is composed of a lower triangular matrix L, a diagonal matrix D, and an upper triangular matrix U. This technique is called the Jacobi method. The matrix addition $L + D + U$ can be written as

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ a_{2,1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & 0 & 0 \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & 0 \end{pmatrix} + \begin{pmatrix} a_{1,1} & 0 & \cdots & 0 & 0 \\ 0 & a_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_{n,n} \end{pmatrix} + \begin{pmatrix} 0 & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ 0 & 0 & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{21}$$

2 Activity

Consider the following matrix:



3 Machine Problem – Circuit of Resistors

Consider the circuitry with resistors all with resistance R and a voltage $V_+ = 5\ V$.

Using Kirchhoff's Junction Rule, the junction at V_1 has the following equation

$$\frac{V_1 - V_2}{R} + \frac{V_1 - V_3}{R} + \frac{V_1 - V_4}{R} + \frac{V_1 - V_+}{R} = 0 \tag{22}$$

which is equivalent to the equation

$$4V_1 - V_2 - V_3 - V_4 = V_+ \tag{23}$$

Determine the three more equations for the junctions at V_2, V_3, and V_4. The resulting system of equations can be used to solve for the values of the four voltages. Use any of the discussed methods to find these voltages.