Applied Physics 158

Root Finding Methods

1^{st} sem, A.Y. 2020–2021

1 Methods

The problem is from the given function $f(x)$, determine the root x^* such that

$$f(x^*) = 0 \tag{1}$$

Numerically, this equation may be approximated using a stopping tolerance ϵ.

$$f(x^*) < \epsilon \tag{2}$$

where ϵ is close enough to zero.

1.1 Bisection Method

The simplest method in root finding uses an interval $[a, b]$ in the domain of the function where the bounds of the interval satisfy:

$$f(a)f(b) < 0 \tag{3}$$

or in other words, $f(a)$ and $f(b)$ have opposing signs. The successive process finds the midpoint c bisecting the interval:

$$c = \frac{a+b}{2} \tag{4}$$

and tests if on that iteration, $f(c) < \epsilon$ which implies that c is a root of f. On the other hand, if $f(c) \not< \epsilon$, then the bounds are reassigned. If $f(c)$ has the similar sign with $f(a)$, the next iteration uses the interval $[c, b]$ and if $f(c)$ has the similar sign with $f(b)$, the next iteration uses the interval $[a, c]$.

Note that this method requires that the function is continuous on the interval $[a, b]$.

1.2 Newton's Method

This method requires continuity as well since the derivative of the function is needed. Additionally, it also requires an initial guess x_0 and repetitively determines the root using the formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{5}$$

and tests if on that iteration, $f(x_{i+1}) < \epsilon$ which implies that x_{i+1} is a root of f.

1.3 Secant Method

Regarded as a *quasi*-Newton technique, this method is related to the previous one since

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \tag{6}$$

and the difference between the two methods is that the secant method requires two initial guesses. Substitute this to the previous equation,

$$x_{i+1} \approx x_i - \frac{f(x_i)}{\left(\frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}}\right)} \tag{7}$$

which can be rewritten as

$$x_{i+1} \approx x_i - f(x_i)\left(\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}\right) \tag{8}$$

If on that iteration, $f(x_{i+1}) < \epsilon$, then x_{i+1} is a root of f.

2 Activity

Consider the following function

$$f(x) = x^6 - x^5 - x^4 - x^3 - x^2 - x \tag{9}$$

and choose ONE method to analytically find the roots of this function. For all methods, use the stopping criterion $\epsilon = 10^{-4}$.
Bisection method:

1. Use the bisection method on the interval $[-1, 1]$.
2. Plot each iteration i versus the value of c_i until convergence.
3. Similarly, use the bisection method on the interval $[1.5, 6]$ and also plot each iteration i versus the value of c_i until convergence.
4. The roots of the function are 0 and 1.96595. Compare them with what you determined numerically.
5. What is the difference between using the two intervals for the bisection method?

Newton method:

1. Use the Newton or secant method with the initial point $x_0 = 1.55$.
2. Plot each iteration i versus the value of x_i until convergence.
3. Similarly, use the Newton or secant method with the initial point $x_0 = 1.60$ and also plot each iteration i versus the value of x_i until convergence.
4. The roots of the function are 0 and 1.96595. Compare them with what you determined numerically.
5. What is the difference between using the two initial points for the Newton or secant methods?

Secant method:

1. Use the secant method with the initial points $x_0 = -2$ and $x_1 = 1.5$.
2. Plot each iteration i versus the value of x_i until convergence.
3. Similarly, use the secant method with the initial points $x_0 = 1.5$ and $x_1 = 2.25$ and also plot each iteration i versus the value of x_i until convergence.
4. The roots of the function are 0 and 1.96595. Compare them with what you determined numerically.
5. What is the difference between using the two intervals for the bisection method?

3 Machine Problem – The Lagrange point

The L_1 Lagrange point is the perfect point collinear between the Earth and the moon where satellites stay due to the inward pull of the Earth and the outward pull of the moon. Its equation is given by:

$$\frac{GM}{r^2} - \frac{Gm}{(R-r)^2} = \omega^2 r \tag{10}$$

where M is the mass of the Earth, m is the mass of the moon, G is the gravitational constant, R is the distance from Earth to moon, and ω is the angular velocity of the moon and the satellite.

Given the following constants:

- $M = 5.974 \times 10^{24}\ kg$
- $m = 7.348 \times 10^{22}\ kg$
- $G = 6.674 \times 10^{-11}\ \frac{m^3}{kgs^2}$
- $R = 3.844 \times 10^{8}\ m$
- $\omega = 2.662 \times 10^{-6} \frac{1}{s}$

Solve for the variable r which is the distance from Earth to the satellite. Choose a suitable method and initial conditions.