Section A

1. (a) The sizes of two interacting populations, $x_1(t)$ and $x_2(t)$, are modelled over a short time frame (with time measured in years) by the linear system of differential equations

$$\vec{x}\,' = A\vec{x}, \quad \text{where} \quad A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

   (i) Solve this system of differential equations along with the initial conditions

$$\vec{x}(0) = \begin{bmatrix} 4000 \\ 1000 \end{bmatrix}.$$

[7 marks]

   (ii) If this model is only valid up to the first positive time t_{end} such that $x_1(t_{end}) = 1\ 000$, find the value of t_{end} rounded to two decimal places.
   HINT: it might be helpful to use technology (e.g. MATLAB/Octave or Python) to help answer this question.

[2 marks]

   (b) The weight, $w(t)$ in grams (g), of a starving animal t days after it stops eating is modelled by the differential equation

$$\frac{dw}{dt} = -\frac{1}{3}w^{\frac{4}{5}}.$$

   (i) Solve this differential equation for w as an explicit function of t in the special case of an animal whose weight at the start of starving is 100 000 g, then calculate the weight of that animal after 10 days of starving, giving your answer rounded to the nearest whole gram.

[4 marks]

   (ii) *Chossat's law* says that an animal dies after it has lost 50% of it body weight. Under the assumption that *Chossat's law* applies, how long would it take for the animal described in part (i) to die? **Do not answer this question by trial-and-error. Show all calculations and round your answer to the nearest whole day**.

[2 marks]

[15 MARKS TOTAL]

2. A model for population growth of a species of fish in a lake is given by

$$\frac{dx}{dt} = f(x) = ax(M-x)+B = aMx\left(1-\frac{x}{M}\right)+B$$

where a, M, and B are positive parameters and $x(t)$ is the population of fish in the lake at time t months after the initial time.

(a) Give a quick description of what type of growth the equation models, including explaining what B could represent and what are its units of measurement.

[3 marks]

(b) Find ALL equilibrium solutions of this equation and use calculus to classify each as stable, semistable, or unstable.

[8 marks]

(c) Say, with reason, which equilbrium solution is never relevant for this population problem *(you will receive no marks if you do not give a reason for your answer)*.

[1 mark]

(d) Confirm your classification of the equilibrium solutions in the special case $a = 0.001$, $M = 40$, and $B = 4$ by producing a suitable direction field plot for $t \in [0, 10]$ and the range of x values chosen so that the direction field shows ALL the equilibrium solutions. Clearly state what the equilibrium solutions are and classify them using the direction field, stating clearly what aspect of the direction field allows you to conclude whether an equilibrium solution is stable, semistable, or unstable.

[4 marks]

(e) Follow the procedure outlined in **Lecture 5** to express the differential equation

$$\frac{dx}{dt} = f(x) = ax(M-x)+B = aMx\left(1-\frac{x}{M}\right)+B$$

in non-dimensional terms so that there remains only one dimensionless parameter b and the equation becomes

$$\frac{du}{d\tau} = u(1-u)+b$$

where x has been replaced with a non-dimensional u and t replaced with a non-dimensional τ. State clearly what the new dimensionless variables u and τ are in terms of the old variables and parameters and what the new dimensionless parameter b is in terms of the old parameters.

[4 marks]

[20 MARKS TOTAL]

3. The following system of differential equations models interaction between two species:

$$\frac{dx}{dt} = x\left(\frac{y}{x+1} - 1\right), \qquad \frac{dy}{dt} = y\left(\frac{x}{y+1} + 10 - y\right),$$

where $x(t)$ is a measure of the population of one species and $y(t)$ is a measure of the population of the other species at time t. All references in the rest of this question to *this system of equations* or *the given system* mean the system of equations given above.

(a) Compare this system of equations to the Lotka-Volterra predator-prey equations

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= -cy + dxy \end{aligned}$$

where $x(t)$ is the prey population and $y(t)$ is the predator population at time t. Specifically, state what is a crucial difference between the "encounter terms" (the terms involving the product xy) in the given system compared to the encounter terms in Lotka-Volterra system, and explain why this difference rules out the possibility of the given system of equations modelling predator-prey interaction.

[2 marks]

(b) Find all three steady states of this system of equations where both $x \geq 0$ and $y \geq 0$ and classify them (stability AND type of steady state, for example "asymptotically stable spiral point"), using calculus and linear algebra.

[12 marks]

(c) Solve this system of equations using at least a second order numerical method (so Heun's method, the Runge-Kutta method, MATLAB/Octave's in-built `ode45`, or Python's `scipy.integrate.ivp_solver` are acceptable but not Euler's method) over the interval $t \in [0, 20]$, using $\Delta t = 0.25$. State clearly which approximation method you use to solve the system and, specifically, do the following:

(i) Experiment with a range of initial conditions in which $x(0) = 0$ and $y(0) > 0$. What do you see in the plots of x versus t and y versus t?

[2 marks]

(ii) Experiment with a range of initial conditions in which $x(0) > 0$ and $y(0) = 0$. What do you see in the plots of x versus t and y versus t? Based on your observations so far, say with reason which species seems more robust.

[3 marks]

(iii) Experiment with a range of initial conditions in which $x(0) > 0$ and $y(0) > 0$. What do you see in the plots of x versus t and y versus t? Submit plots of x versus t and y versus t for the case $x(0) = 8$ and $y(0) = 7$ AND also submit a direction field plot with x values between 0 and 15 and y values between 0 and 15. Discuss how the direction field plot explains at least one of the results from your numerical experiments in parts (i), (ii), or earlier in (iii).

[4 marks]

(d) Explain why the given system of equations is more likely to model mutualism instead of competition.

[2 marks]

[25 MARKS TOTAL]